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Threshold dynamics in hyperbolic partial differential equations

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Threshold dynamics in hyperbolic partial differential equations

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ABSTRACT

We are interested in the persistence of the C^1 solution regularity for time dependent partial differential equations(PDE). As is known that the typical well-posedness result asserts that either a solution of a time-dependent PDE exists for all time or else there is a finite time such that some norm of the solution becomes unbounded as the life span is approached. The natural question is whether there is a critical threshold for the initial data such that the persistence of the C^1 solution regularity depends only on crossing such a critical threshold. In this thesis we attempt to study such a critical phenomena in Restricted Euler-Poisson(REP) equations and a class of non-local conservation laws.

In this thesis, we have obtained the following results.

1. For three-dimensional REP equations, we identify both upper thresholds for the finite-time blow up of solutions and subthresholds for the global existence of solutions, with the thresholds depending on the relative size of the eigenvalues of the initial velocity gradient matrix and the initial density. For the attractive forcing case, these one-sided threshold conditions of the initial configurations are optimal, and the corresponding results also hold for arbitrary n dimensions ($n \geq 3$).

2. We propose weakly restricted Euler-Poisson(WREP) equations as an effort to gain a better understanding on Euler-Poisson equations in multi-dimension. The WREP can be viewed as a slight generalization of the REP equations. We then provide upper-thresholds for finite time blow up of solutions for WREP equations with attractive/repulsive forcing. It is shown that the thresholds depend on the relative size of the initial density and each elements of the initial velocity gradient matrix.

3. We investigate a class of nonlocal conservation laws with the nonlinear advection coupling both local and nonlocal mechanism, which arises in several applications such as the collective motion of cells and traffic flows. It is proved that the C^1 solution regularity of this class of

conservation laws will persist at least for a short time. This persistency may continue as long as the solution gradient remains bounded. Based on this result, we further identify sub-thresholds for finite time shock formation in traffic flow models with Arrhenius look-ahead dynamics. Our threshold analysis for the traffic flow models is applicable to the class of nonlocal conservation laws.

4. Lastly, we further study the class of nonlocal conservation laws. It is well known that the initial value problem for a scalar conservation law may admit more than one weak solution, so we need to find a selection criterion in order to single out the physically relevant solution. We define the Kružkov-type entropy solution, and by adapting the doubling of variables method and the method of vanishing viscosity, we obtain a uniqueness and existence of entropy solutions of the nonlocal conservation laws.

CHAPTER 1. GENERAL INTRODUCTION

The goal of this thesis is to investigate the threshold dynamics in restricted Euler-Poisson equations and a class of nonlocal conservation laws.

1.1 General Background

We are concerned with time regularity of solutions for a class of time dependent partial differential equations(PDE), where the velocity field u is governed by the Newtonian law,

$$\partial_t u + u \cdot \nabla_x u = F, \tag{1.1.1}$$

where $F = F(u, \nabla u, \int K * u)$ is a general forcing acting on the flow. The above Eulerian equations show up in many contexts dictated by the different modeling of F's. For example, Euler-Poisson equations and Navier-Stokes equations.

When it comes to the question of the persistence of time regularity for such equations, as it is known that the typical well-posedness result asserts that either a solution of time-dependent PDE exists for all time or else there is a finite time such that some norm of the solution becomes unbounded as the life span is approached. The natural question is whether there is a critical threshold for the initial data such that the persistence of the solution regularity depends only on crossing such a critical threshold. This concept of critical threshold and associated methodology was originated and developed in a series of papers by Engelberg, Liu and Tadmor [Engelberg et al. (2001); Liu et al. (2002, 2003)] for a class of Euler-Poisson equations.

Let us mention their motivation for introducing a new notion of critical threshold. When dealing with the questions of time regularity for the above Euler-related equations, one encounters several limitations of the classical stability analysis. Indeed, in several references including [Liu et al. (2003), Liu et al. (2003b)], they point out that i) the usual stability analysis does not tell us how large perturbations are allowed before losing stability, e.g. [Kreiss (2000)]; ii) the steady solution may be only conditionally stable due to the weak dissipation in the system [Engelberg et al. (2001)]. In other words, the persistence of the global features of solutions does not fall into any particular category (global smooth solution, finite time breakdown, etc.), but instead, these features depend on crossing a critical threshold associated with the initial configuration of underlying problems - the so called Critical Threshold (CT) phenomena.

To investigate this remarkable CT phenomena, in this introduction section, we start with a simple one-dimensional problem where $F \equiv 0$ in (1.1.1). In case $F \equiv 0$, (1.1.1) is reduced to the unforced inviscid Burgers' equation, where the solution always forms a shock discontinuity except for the case of increasing initial profile, $u_0' \geq 0$, which is non-physical. In the context of the one-dimensional Euler-Poisson equations, however, there is a delicate balance among various forcing mechanisms, which supports a critical threshold phenomena. In particular, consider the basic model with zero background

$$\rho_t + (\rho u)_x = 0, \quad u_t + uu_x = -k\phi_x, \quad -\phi_{xx} = \rho.$$

Here, k is a given constant which signifies the property of the underlying repulsive $k > 0$ or attractive $k < 0$ forcing governed by the Poisson potential. The unknowns are the velocity field $u = u(x,t)$, the density of negatively charged matter $\rho = \rho(x,t)$, and the potential $\phi = \phi(x,t)$. In [Engelberg et al. (2001)], it was shown the equation has global smooth solutions as long as its initial (ρ_0, u_0) configuration satisfies $u_0' \geq -\sqrt{2k\rho_0}$. Here, the critical threshold phenomena is understood as a delicate balance between the forcing mechanism(Poisson equation), and the nonlinear focusing (Newton's second law). Including this milestone work, various issues on CT phenomena for a class of one-dimensional Euler-Poisson equations with or without various forcing mechanisms were investigated by Engelberg, Liu and Tadmor [Engelberg et al. (2001)].

Moving to the multidimensional setup, identifying some proper quantities that describe the critical threshold phenomenon is essential. In [Liu et al. (2002)], Liu and Tadmor show that the proper quantities that describe CT phenomenon depend in an essential manner on the eigenvalues of the velocity gradient matrix $M := \nabla \mathbf{u}$. Indeed, $\lambda = \lambda(\nabla \mathbf{u})$ are shown to be governed by the Riccati type equation $\lambda_t + u \cdot \nabla\lambda + \lambda^2 = \langle l, \nabla F r\rangle$. To illustrate their method of spectral dynamics which relies on the dynamical system governing eigenvalues of M, we consider the following n-dimensional(nD) Euler-Poisson (EP) equations,

$$\begin{aligned} &\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, \\ &\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = k \nabla \Delta^{-1}(\rho - c_b), \end{aligned} \tag{1.1.2}$$

with constant background state $c > 0$. To trace the evolution of M, they differentiate the second equation of (1.1.2), obtaining

$$\partial_t M + \mathbf{u} \cdot \nabla M + M^2 = kR[\rho - c_b],$$

where $R[\cdot]$ is the Risez matrix operator, defined as

$$R[f] := \nabla \otimes \nabla \Delta^{-1}[f].$$

Then, the Euler-Poisson equations are recast into the coupled system

$$M' + M^2 = kR[\rho - c_b], \tag{1.1.3a}$$

$$\rho' + \rho trM = 0, \tag{1.1.3b}$$

with $'$ standing for the usual convective derivative, $\partial_t + \mathbf{u} \cdot \nabla$. The global nature of the Riesz matrix, $R[\rho - c_b]$ makes the issue of regularity for Euler-Poisson equations such an intricate question to solve.

To gain better understanding of the dynamics of the velocity gradient M governed by (1.1.3a)-(1.1.3b), Liu and Tadmor introduce in [Liu et al. (2002)] the restricted Euler-Poisson (REP) system (2.1.3), which is obtained from (1.1.3a) by restricting attention to the local isotropic trace $\frac{k}{n}(\rho - c_b)I_{n\times n}$, of the global coupling term $kR[\rho - c_b]$, namely,

$$M' + M^2 = \frac{k}{n}(\rho - c_b)I_{n\times n}, \tag{1.1.4a}$$

$$\rho' + \rho trM = 0, \tag{1.1.4b}$$

subject to initial data

$$(M, \rho)(0, \cdot) = (M_0, \rho_0).$$

This localization was motivated by the so called restricted Euler equations proposed in [Vieillefosse (1982)] as a localized alternative to the incompressible Euler equation.

For global existence of solutions to the 2D REP system (1.1.4) with $n = 2$, a complete description of the critical threshold criterion was obtained in [Liu et al. (2003)]. In particular, it is shown that the critical thresholds depend on the relative sizes of three quantities: the initial divergence($tr(M_0)$), the initial density(ρ_0), and the initial spectral gap($\Gamma_0 := (\lambda_{20} - \lambda_{10})^2$), that is, the difference between the two eigenvalues of the initial velocity gradient.

To put our study in a proper perspective we recall a few more related references. The Critical Threshold phenomena for one-dimensional or restricted models are investigated in a series of papers by Engelberg, Liu and Tadmor. The key argument of their study for critical thresholds in [Engelberg et al. (2001); Liu et al. (2002, 2003, 2004)] has been based on the convective derivative along particle paths $' = \partial_t + \mathbf{u} \cdot \nabla$, so the threshold results are pointwise and obtained via the Lagrangian approach. In the one-dimensional case, the study of the 2-by-2 ODE solutions of (u_x, ρ) yields the C^1 regularity of the PDE solution. Similar results stay valid for Euler–Poisson systems with geometric symmetry in higher dimensions [Engelberg et al. (2001)]. Beyond these threshold results for one-dimensional or restricted models, effort has been made to extend the Critical Threshold argument to more general models. For the 1D EP system with pressure, Tadmor and Wei [Tadmor et al. (2008)] obtain thresholds through tracking (u_x, ρ) along two characteristic fields. Chae and Tadmor [Chae et al. (2008)] and Cheng and Tadmor [Cheng et al. (2009)] obtain the blow up result for multi-D full Euler–Poisson systems (1.1.2) with attractive forcing $k < 0$. For proofs of the results in [Chae et al. (2008); Cheng et al. (2009)], the vanishing initial vorticity condition which amounts to the symmetry of M is essential.

For full Euler-Poisson system with pressure $p = \rho^\gamma$, $1 < \gamma < 2$, there are several global existence and stability results. [Guo (1998)] constructed global smooth irrotational flows with small perturbed initial data in three-dimensional Euler-Poisson system. [Jang (2011)] showed the existence of global smooth solutions to the two-dimensional Euler-Poisson system for spherically symmetric flows with small perturbed initial data. Many results on stability/instability of stationary solutions can be found in [Lin (1997); Deng et al. (2002); Rein (2003); Jang (2008)]

In this thesis we investigate the CT phenomenon for the 3D REP system (1.1.4), as well as the nD REP system. Including a summary of main results, the main difficulties and our main ideas in investigating the CT phenomenon for multi-dimensional REP system will be discussed in a separate section later in this chapter.

Next we extend our discussion of the CT phenomena to a class of nonlocal conservation laws, i.e. if $F = -\nabla_x \cdot G(u, \bar{u}) + u \cdot \nabla_x u$ then the equation in ((1.1.1)) corresponds to

$$\begin{cases} \partial_t u + \partial_x G(u, \bar{u}) = 0, & t > 0, x \in \mathbb{R}^n, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.1.5}$$

Here, G is a given smooth function, and $\bar{u}$ is given by

$$\bar{u}(t, x) = (K * u)(t, x) = \int_{\mathbb{R}^n} K(x-y)u(t, y)\, dy,$$

where K is a given kernel. The advection couples both local and nonlocal mechanisms. This class of conservation laws appears in several applications including traffic flows [Kurganov et al. (2009); Sopasakis et al. (2006)], the collective motion of biological cells [Dolak et al. (2005); Burger et al. (2008); Perthame et al. (2009)], dispersive water waves [Whitham (1974); Holm et al. (2005); Degasperis et al. (1999); Liu (2006)], the radiating gas motion [Hamer (1971); Rosenau (1989); Liu et al. (2001)], high-frequency waves in relaxing medium [Hunter (1990); Parkes (2002); Vakhnenko (1992)], and the kinematic sedimentation [Kynch (1952); Zumbrun (1999); Karlsen et al. (2011)].

The traffic flow model that motivated this study is the one with look-ahead relaxation introduced by Sopasakis and Katsoulakis [Sopasakis et al. (2006)]:

$$\begin{cases} \partial_t u + \partial_x (u(1-u)e^{-K*u}) = 0, & t > 0, x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1.6}$$

where $u(t, x)$ represents a vehicle density normalized in the interval $[0, 1]$ and the relaxation kernel

$$K(r) = \begin{cases} \frac{K_0}{\gamma}, & \text{if } -\gamma \le r \le 0, \\ 0, & \text{otherwise}, \end{cases} \tag{1.1.7}$$

is the constant interaction potential, where γ is a positive constant proportional to the look-ahead distance and K_0 is a positive interaction strength. We set $K_0 = 1$ since in our study this parameter is not essential. The finite time shock formation of solutions in traffic flows are understood as congestion formation.

An improved interaction potential for (1.1.6) is introduced in [Kurganov et al. (2009)] with

$$K(r) = \begin{cases} \frac{2}{\gamma}\left(1+\frac{r}{\gamma}\right), & -\gamma \leq r \leq 0, \\ 0, & \text{otherwise}. \end{cases} \tag{1.1.8}$$

This linear potential is intended to take into account the fact that a car's speed is affected more by nearby vehicles than distant ones. The authors in [Kurganov et al. (2009)] carried out some careful numerical study of the traffic flow model (1.1.6), through three examples: red light traffic, traffic jam on a busy freeway and a numerical breakdown study. In the case of a good visibility (large γ), their numerical studies suggest that (1.1.6) with the modified potential (1.1.8) yields solutions that seem to better correspond to reality.

Recently D. Li and T. Li presented several finite time shock formation scenarios of solutions to (1.1.6) with (1.1.7). Their approach is to analyze the solutions along two characteristic lines defined by $0 = u(t, X_1(t))$ and $1 = u(t, X_2(t))$, with which they justified that if there exist two points $\alpha_1 < \alpha_2$, such that $u_0(\alpha_1) = 0$ and $u_0(\alpha_2) = 1$, then u_x must blow up at some finite time.

In this thesis we investigate the threshold phenomenon for the traffic flow models with Arrhenius look-ahead dynamics, as well as a class of non-local conservation laws. Including a summary of main results, the main difficulties and our main ideas in identifying sub-thresholds for finite time shock formation in the traffic flow models will be discussed in a separate section.

We further discuss non-local conservation laws (1.1.5) in the context of weak solutions. Already in the example on traffic flow, it may be the case that characteristics cross at the point at which two densities and the characteristics merge into a shock. This leads to a density discontinuity. To solve problems with discontinuities in density, Lax introduced weak solutions that satisfy the conservation law in its integral form. However, such weak solutions are not unique. The problem of lack of uniqueness for weak solutions is intrinsic in the theory of conservation laws. There are several different approaches to this problem, the common notion is the so-called entropy condition.

To put our study in a proper perspective we recall a few related references. Entropy conditions were first studied independently by [Lax (1957)] and [Oleinik (1957)], and the uniqueness and existence of the entropy solution are proved by [Kružkov (1970)]. Conservation law related references are the following: [Lax (1971)] first recommended the addition of an entropy condition to help select a physically relevant solution from a set of weak solutions in gas dynamics. [Ansorge (1990)] pointed out that the entropy condition in gas dynamics can be used in traffic flow theory as a uniqueness criterion. [Karlsen et al. (2011)] defined the Kružkov entropy solutions for nonlocal conservation laws modeling sedimentation, and show uniqueness and existence.

Here we provide a brief motivation for using the Kružkov type entropy condition [Kružkov (1970)]. Indeed, this condition is often more convenient to work with in the sense that it combines the definition of a weak solution with that of the entropy condition. We consider the viscous regularization of the standard conservation law,

$$\partial_t u^\epsilon + \partial_x f(u^\epsilon) = \epsilon \partial_x^2 u^\epsilon, \tag{1.1.9}$$

as $\epsilon \searrow 0$. The Kružkov type condition follows from a standard vanishing viscosity argument. Roughly speaking, the idea is based on the demand that the distributional solution of the standard conservation laws should be limits of solutions of the more fundamental equation (1.1.9) as the viscous term disappears. We choose a smooth convex function $\eta = \eta(u)$ and a non-negative test function $\phi \in C_0^\infty(\mathbb{R} \times [0,\infty])$. Then, from (1.1.9), we find

$$0 = \int_0^\infty \int_{\mathbb{R}} (\partial_t u + \partial_x f(u^\epsilon) - \epsilon \partial_x^2 u^\epsilon)\eta'(u)\phi \, dxdt \geq - \int_0^\infty \int_{\mathbb{R}} (\eta(u)\phi_t + q(u)\phi_x + \epsilon \eta(u)\phi_{xx}) \, dxdt,$$

where q is such that

$$q'(u) = f'(u)\eta'(u).$$

As $\epsilon \searrow 0$, we have

$$\int_0^\infty \int_{\mathbb{R}} \eta(u)\phi_t + q(u)\phi_x \, dxdt \geq 0. \tag{1.1.10}$$

If we let $\eta(u) = |u-k|$ for some constant k, we find that

$$q(u) = sgn(u-k)(f(u) - f(k)).$$

It turns out that if (1.1.10) holds for $\eta(u) = |u-k|$ for all $k \in \mathbb{R}$, then this inequality holds for all convex functions. This celebrated entropy-entropy flux pair $(|u-k|, sgn(u-k)(f(u)-f(k)))$ gives us the motivation of defining the entropy solutions for the non-local conservation laws. In this thesis we discuss a uniqueness and existence result of the entropy solutions.

1.2 Thesis Organization

The thesis is organized as follows. In Chapter 2, the multi-dimensional restricted Euler-Poisson equation is investigated and we identify both upper-thresholds for finite time blow up of solutions and sub-thresholds for global existence of solutions. In Chapter 3, weakly restricted Euler-Poisson equations are proposed to gain a better understanding on Euler-Poisson equations in multi-dimensions. We then provide upper-thresholds for finite time blow up of solutions for WREP. Chapter 4 is reserved for the investigation of a class of nonlocal conservation laws and the identification of threshold conditions for finite time shock formation in traffic flow models. In Chapter 5, we further investigate the class of nonlocal conservation laws. The Kružkov-type entropy solution concept is introduced and we discuss the uniqueness and existence of entropy solutions. The content of each chapter is summarized in the following four subsections.

1.3 Thresholds in Three-dimensional Restricted Euler-Poisson Equations

In this thesis we first investigate the n-dimensional REP system. Our results reveal threshold conditions on the initial data that lead to the finite time blow up or global boundedness of M.

The main difficulty in investigating the CT phenomenon for the nD REP system is that because of its multidimensional nature, too many quantities are needed to be controlled. Furthermore, in [Liu et al. (2003)], it was pointed out that identifying the CT phenomenon even for 3D REP system is a formidable task due to this difficulty. We resolve this obstacle by adapting the spectral dynamics approach taken in [Liu et al. (2003)] to the nD REP system, which leads to a closed $(n+1)\times(n+1)$ nonlinear system of ODEs governing the time dynamics of the nD REP. Another difficulty is that with this $(n+1)\times(n+1)$ nonlinear system of ODEs, it is no longer possible to employ the precise phase plane analysis as carried out in the previous works. To attack this difficulty, we use the order preserving property of eigenvalues of $M = \nabla \mathbf{u}$ and obtain 2×2 ODE system with time-dependent coefficients. Finally, delicate comparison with relatively simple ODE systems enable us to obtain the desired results.

The main results are summarized as follows. Without loss of generality, we shall label the initial eigenvalues

$$\lambda(M) = \{\lambda_i\}_{i=1}^n$$

in terms of the real part of each eigenvalue such that

$$\mathrm{Re}(\lambda_{10}) \leq \mathrm{Re}(\lambda_{20}) \leq \cdots \leq \mathrm{Re}(\lambda_{n0}).$$

For the nD REP system (1.1.4) with nonzero background $c_b > 0$ and initial density $\rho_0 > 0$, we have the following.

- (Attractive case $k < 0$) If λ_{10} is real, and there exists $\Lambda_n(k, \rho_0)$ such that

  $$\lambda_{10} > \Lambda_n(k, \rho_0), \quad n \geq 3,$$

  then the solution remains bounded for all time.

  If all $\{\lambda_{i0}\}_{i=1}^n$ are real, and

  $$\lambda_{n0} < \Lambda_n(k, \rho_0),$$

  then the solution will blow up in finite time.

- (Repulsive case $k > 0$) Suppose that all eigenvalues are initially real. The solution remains bounded for all time if all eigenvalues are initially identical.

  If the spectral gap

  $$\lambda_{20} - \lambda_{10} > \Gamma_n(k, \rho_0),$$

  where Γ_n denotes the gap thresholds, then the solution of the nD REP system will blow up in finite time for $n = 3, 4$.

The results are more precisely stated in Chapter 2, together with relevant remarks.

1.4 Finite Time Blow-up of Solutions to 2D Weakly Restricted Euler-Poisson Equations

Next we propose weakly restricted Euler-Poisson (WREP) system as a semi-localized alternative to (1.1.4). Specifically, we consider a system for (ρ, M), governed by

$$\frac{d}{dt}M + M^2 = \begin{pmatrix} \frac{k}{2}(\rho - c_b) & R_{12} \\ R_{21} & \frac{k}{2}(\rho - c_b) \end{pmatrix},$$
$$\frac{d}{dt}\rho + \rho \mathrm{tr} M = 0,$$

subject to initial data

$$(M, \rho)(0, \cdot) = (M_0, \rho_0).$$

This differs from (1.1.4) in the sense that non-local terms R_{12}, R_{21} are still kept. We proceed to investigate threshold conditions on the initial data that lead to the finite time blow up of M and ρ.

1.5 Thresholds for Shock Formation in Traffic Flow Models with Arrhenius Look-ahead Dynamics

We extend our discussion of the threshold phenomena to a class of non-local conservation laws. We first show that C^1 solution regularity of (1.1.5) persists at least for finite time. Moreover, such persistency may continue as long as the solution gradient remains bounded. Our procedure to prove the local existence is summarized as follows:

1. Apply the Banach fixed-Point theorem to the transformation S defined through $v = S(u)$, where v is solved from

$$\begin{cases} \partial_t v + G_u v_x + F_{\bar{u}} \bar{u}_x = 0, \\ v(t = 0) = u_0. \end{cases}$$

2. Show that there exists $T > 0$ depending on initial data such that the mapping $v = S(u)$ exists and is a contraction.

3. Detailed estimates of non-local terms are crucial, and allow us to track the dependence of T on the initial profile $\|u_0\|_{H^2}$.

Next we identify thresholds for finite time shock formation in traffic flow models with Arrhenius look-ahead dynamics, as well as (1.1.5) with one sided interaction kernels. Our procedure includes the following crucial ingredients:

1. Trace the Lagrangian dynamics of $d := u_x$, which can be obtained from the Eulerian formulation:

$$(\partial_t + (1-2u)e^{-\bar{u}}\partial_x)d = e^{-\bar{u}}[2d^2 + 2(1-2u)\bar{u}_x d - u(1-u)\bar{u}_x^2 + u(1-u)\bar{u}_{xx}].$$

2. The right hand side is quadratic in d, the a priori bound $0 \leq u \leq 1$ ensures the boundedness of both u and $\bar{u}_x$ involved in the coefficients.

3. The key in our approach is to bound the nonlocal term $\bar{u}_{xx}$ in terms of $M = \sup_{x\in\mathbb{R}}[u_x(x,t)]$ and $N = \inf_{x\in\mathbb{R}}[u_x(x,t)]$ attained at $x = \xi(t)$ and $x = \eta(t)$, respectively.

4. This way we are able to obtain weakly coupled differential inequalities for both M and N, which yield the desired threshold conditions.

Let us point out that this non-standard approach of tracing the dynamics d along two different curves originates in an idea of Seliger [Seliger (1968)] while proving wave breaking for the Whitham equation. To carry out Seliger's formal analysis, one needs to assume that the curves $\xi(t)$ and $\eta(t)$ are smooth. This additional strong assumption was shown unnecessary later by Constantin and Escher [Constantin et al. (1998)].

The above procedure enables us to identify sub-thresholds for finite time shock formation in traffic flow models with the look-ahead dynamics. Our threshold analysis for the traffic flow models is applicable to the class of non-local conservation law under the assumptions that the interaction kernel is one-sided, together with some technical assumptions on G. The main results are summarized as follows.

- If $u_0 \in H^2$ and $0 \leq u_0(x) \leq m$ for all $x \in \mathbb{R}$, then there exists a non-increasing function $\lambda(\cdot)$ such that if

$$\sup_{x\in\mathbb{R}}[u_0'(x)] > \lambda(\inf_{x\in\mathbb{R}}[u_0'(x)]),$$

  then u_x must blow up at some finite time.

We should point out that compared to the recent work of T. Li and D. Li [Li et al. (2011)], our shock formation conditions may be viewed in the perspective of critical thresholds. Furthermore, our shock formation conditions are consistent with the numerical results obtained in [Kurganov et al. (2009)]. The results are more precisely stated in Chapter 4, together with relevant remarks.

1.6 Well-posedness of The Global Entropy Solution to A Class of Nonlocal Conservation Laws

We further investigate non-local conservation laws (1.1.5). It is the purpose of this section to study the well-posedness of equation (1.1.5). It is well known that solutions to a standard nonlinear conservation law are in general discontinuous even if the initial data is smooth, so we need to define solutions as weak solutions. Since weak solutions of conservation laws are, in general, not unique, a selection criterion must be imposed to single out the physically relevant solution. The main part of selection criterion is based on the following Kružkov-type inequality: For all nonnegative test functions $\psi \in C_0^\infty(S_T)$,

$$\iint_{S_T} \partial_t\psi \cdot |u-k| + sgn(u-k)\big[F(u,\bar{u}) - F(k,\bar{u})\big]\psi_x - sgn(u-k)F(k,\bar{u})_x\psi \, dxdt \geq 0,$$

$\forall k \in \mathbb{R}$. Here $S_T = \mathbb{R} \times (0,T)$, $\forall T > 0$.

The main results of this section are the uniqueness and existence of entropy solutions. The uniqueness is proved by adapting the so called doubling of variables method of Kružkov. In the process of adapting the doubling of variables method, because of the presence of the non-local term, careful decompositions are introduced to obtain the desired estimates. The existence result is obtained by using the method of vanishing viscosity.

CHAPTER 2. THRESHOLDS IN THREE-DIMENSIONAL RESTRICTED EULER-POISSON EQUATIONS

A paper published by Physica D: Nonlinear Phenomena

Yongki Lee and Hailiang Liu

Abstract This work provides a description of the critical threshold phenomenon in multi-dimensional restricted Euler-Poisson (REP) equations, introduced in [H. Liu and E. Tadmor, *Comm. Math. Phys.* 228 (2002), 435–466]. For three-dimensional REP equations, we identified both upper-thresholds for finite time blow up of solutions and sub-thresholds for global existence of solutions, with thresholds depending on the relative size of the eigenvalues of the initial velocity gradient matrix and the initial density. For attractive forcing case, these one-sided threshold conditions of initial configurations are optimal, and the corresponding results also hold for arbitrary n-dimensions ($n \geq 3$).

2.1 Introduction

We are concerned with the critical threshold phenomenon in multi-dimensional Euler-Poisson equations. In this paper we consider a localized version of the following n-dimensional(nD) Euler-Poisson (EP) equations,

$$\begin{aligned} &\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, \\ &\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = k \nabla \Delta^{-1}(\rho - c_b), \end{aligned} \tag{2.1.1}$$

which govern the unknown local density $\rho = \rho(t, x)$ and velocity field $\mathbf{u} = \mathbf{u}(t, x)$ subject to initial conditions $\rho(0, x) = \rho_0(x)$ and $\mathbf{u}(0, x) = \mathbf{u}_0(x)$. They involve two constants: the constant k which signifies the property of the underlying repulsive $k > 0$ or attractive $k < 0$ forcing, governed by the Poisson potential $\Delta^{-1}(\rho - c_b)$, and constant $c_b > 0$ which denotes background state. This hyperbolic system (2.1.1) with non-local forcing describes the dynamic behavior of many important physical flows, including charge transport [Markowich et al. (1990)], plasma with collision [Jackson et al. (1975)], cosmological waves [Brauer et al. (1994)] and expansion of cold ions [Holm et al. (1981)], as well as the collapse of stars due to self gravitation ($k < 0$) [Makino, T. (1986); Brenner et al. (1998); Deng et al. (2002)].

There is a considerable amount of literature available on the solution behavior of Euler-Poisson equations. Let us mention the study of steady state solutions, e.g., [Makino, T. (1986); Gamba (1992); Degond et al. (1993); Luo et al. (2004, 2008); Rein (2003)]; the global existence of weak solutions [Chen et al. (1996); Zhang (1995); Marcati et al. (1995); Poupaud et al. (1995)]. Global existence due to damping relaxation and with nonzero background can be found in e.g. [Wang (2001); Wang et al. (1998); Luo et al. (1999)].

For the question of global behavior of strong solutions, however, the choice of the initial data and/or damping forces is decisive. With a repulsive force $k > 0$, we refer to [Guo (1998); Cordier et al. (2000)] for the global existence of classical solutions with small data, and [Perthame (1990)] for the nonexistence of global solutions; With attractive force $k < 0$, we refer to [Makino (1992); Makino et al. (1990)] for nonexistence results. These results rely on some energy methods using small or large enough initial energy.

The nonlocal forcing in (2.1.1) dictated by the Poisson potential is only weakly dissipative, as a result, the steady state may be only conditionally stable. Indeed, for a class of one-dimensional Euler-Poisson equations and multi-dimensional equations with spherical symmetry, it was shown in [Engelberg et al. (2001)] that the persistence of the global features of the solutions hinges on a delicate balance between the nonlinear convection and the non-local forcing. In other words, the persistence of the global features of solutions does not fall into any particular category (global smooth solution, finite time breakdown, etc.), but instead, these features depend on crossing a critical threshold associated with the initial configuration of underlying problems – so called the Critical Threshold (CT) phenomena. The study of such a remarkable CT phenomena opens a new avenue to address the fundamental question of persistence of the C^1 solution regularity for the EP system and related models.

The concept of Critical Threshold and associated methodology is originated and developed in a series of papers by Engelberg, Liu and Tadmor [Engelberg et al. (2001)], Liu and Tadmor [Liu et al. (2002, 2003, 2004, 2010)] and more.

It first appears in [Engelberg et al. (2001)] regarding pointwise criteria for C^1 solution regularity of 1D EP system. The critical threshold obtained therein describes the conditional stability of the 1D EP systems, where the answer to the question of global vs. local existence depends on whether the initial data crosses a critical threshold.

Moving to the multi-dimensional setup, one has to identify the proper quantities to describe the critical threshold phenomenon. Liu and Tadmor introduce in [Liu et al. (2002)] the method of spectral dynamics which relies on the dynamical system governing eigenvalues of the velocity gradient matrix, $M := \nabla \mathbf{u}$, along particle paths. To illustrate this, we differentiate the second equation of (2.1.1), obtaining formally

$$\partial_t M + \mathbf{u} \cdot \nabla M + M^2 = kR[\rho - c_b],$$

where $R[\cdot]$ is the Risez matrix operator, defined as

$$R[f] := \nabla \otimes \nabla \Delta^{-1}[f].$$

Now, Euler-Poisson equations are recast into the coupled system

$$M' + M^2 = kR[\rho - c_b], \tag{2.1.2a}$$

$$\rho' + \rho tr M = 0, \tag{2.1.2b}$$

with $'$ standing for the usual convective derivative, $\partial_t + \mathbf{u} \cdot \nabla$. The global nature of the Riesz matrix, $R[\rho - c_b]$ makes the issue of regularity for Euler-Poisson equations such an intricate question to solve.

To gain better understanding of the dynamics of the velocity gradient M governed by (2.1.2a)-(2.1.2b), Liu and Tadmor introduce in [Liu et al. (2002)] the restricted Euler-Poisson (REP) system (2.1.3), which is obtained from (2.1.2a) by restricting attention to the local isotropic trace $\frac{k}{n}(\rho - c_b)I_{n\times n}$, of the global coupling term $kR[\rho - c_b]$.

The REP system is given by

$$M' + M^2 = \frac{k}{n}(\rho - c_b)I_{n\times n}, \tag{2.1.3a}$$

$$\rho' + \rho trM = 0, \tag{2.1.3b}$$

subject to initial data

$$(M, \rho)(0, \cdot) = (M_0, \rho_0).$$

This localization was motivated by the so called restricted Euler equations proposed in [Vieillefosse (1982)] as a localized alternative to the impressible Euler equation.

For global existence of solutions to 2D REP system (2.1.3) with $n = 2$, a complete description of the critical threshold criterion was obtained in [Liu et al. (2003)]. Beyond the pointwise threshold results obtained in [Engelberg et al. (2001); Liu et al. (2002, 2003, 2004)] for one-dimensional or restricted models, effort has been made to extend the Critical Threshold argument to more general models.

For the 1D EP system with pressure, Tadmor and Wei [Tadmor et al. (2008)] obtain thresholds through tracking (u_x, ρ) along two characteristic fields. Chae and Tadmor [Chae et al. (2008)] obtain the blow up result for multi-D full Euler–Poisson systems (2.1.3) with attractive forcing $k < 0$. Cheng and Tadmor [Cheng et al. (2009)] obtained (2.2.9), which improved the result of [Chae et al. (2008)]. For proofs of the results in [Chae et al. (2008); Cheng et al. (2009)], the vanishing initial vorticity condition which amounts to the symmetry of M is essential to ensure the key inequality (2.2.8).

In this work we further investigate the 3D REP system (2.1.3), as well as the nD REP system. Our results reveal threshold conditions on the initial data that lead to the finite time blow up or global boundedness of M. They quantify the balance between density ρ and eigenvalues $\lambda(M) = \{\lambda_i\}_{i=1}^n$. Without loss of generality, we shall label the initial eigenvalues in terms of the real part of each eigenvalue such that

$$\mathrm{Re}(\lambda_{10}) \leq \mathrm{Re}(\lambda_{20}) \leq \cdots \leq \mathrm{Re}(\lambda_{n0}).$$

The main results are summarized as follows: For the nD REP system (2.1.3) with nonzero background $c_b > 0$ and initial density $\rho_0 > 0$, we have the following.

- (Attractive case $k < 0$) If λ_{10} is real, and there exists $\Lambda_n(k, \rho_0)$ such that

$$\lambda_{10} > \Lambda_n(k, \rho_0), \quad n \geq 3,$$

then the solution remains bounded for all time; If all $\{\lambda_{i0}\}_{i=1}^n$ are real, and

$$\lambda_{n0} < \Lambda_n(k, \rho_0),$$

then the solution will blow up in finite time.

- (Repulsive case $k > 0$) Suppose that all eigenvalues are initially real. The solution remains bounded for all time if all eigenvalues are initially identical; If the spectral gap

$$\lambda_{20} - \lambda_{10} > \Gamma_n(k, \rho_0),$$

where Γ_n denotes the gap thresholds, then the solution of the nD REP system will blow up in finite time for $n = 3, 4$.

These results are more precisely stated in section 2: Theorem 2.2.1-2.2.2 ($n = 3$) and Theorem 2.2.7-2.2.8 ($n > 3$) for $k < 0$; Theorem 2.2.3-2.2.4 ($n = 3$) and Theorem 2.2.9-2.2.10 ($n > 3$) for $k > 0$, together with relevant remarks.

In section 3, we prove both global existence and finite time blow up of solutions to the REP system with attractive forcing. In section 4 we study the thresholds for the REP system with repulsive forcing. Extension to n-dimensional case is carried out in section 5.

2.2 Statement of Main Results

We first present results which quantify the balance between density ρ and eigenvalues $\lambda(M) = \{\lambda_i\}_{i=1}^3$. These results, as a generalization of those in [Liu et al. (2003)], also hold in arbitrary dimensions ($n > 3$) when $k < 0$, for which further discussion is given after the statement of the 3D theorems.

Theorem 2.2.1. (Global existence for 3D REP with k < 0). *Consider the 3D attractive REP system, (2.1.3) with $k < 0$ and with nonzero background $c_b > 0$. If $\lambda_{10} \in \mathbb{R}$, then the solution of the 3D REP system remains bounded for all time provided $\rho_0 > 0$ and*

$$\lambda_{10} > sgn(\rho_0 - c_b)\sqrt{k\left(c_b^{\frac{1}{3}}\rho_0^{\frac{2}{3}} - \frac{2}{3}\rho_0 - \frac{1}{3}c_b\right)}. \tag{2.2.1}$$

Theorem 2.2.2. (Finite time blow up for 3D REP with k $<$ 0). *Consider the* $3D$ *attractive REP system, (2.1.3) with* $k<0$ *and with nonzero background* $c_b>0$. *Assume* $\lambda(M_0)\in\mathbb{R}$. *The solution of the* $3D$ *REP system will blow up in finite time if* $\rho_0>0$ *and*

$$\lambda_{30} < sgn(\rho_0 - c_b)\sqrt{k\left(c_b^{\frac{1}{3}}\rho_0^{\frac{2}{3}} - \frac{2}{3}\rho_0 - \frac{1}{3}c_b\right)}. \tag{2.2.2}$$

Theorem 2.2.3. (Global existence for 3D REP with k $>$ 0). *Consider the* $3D$ *repulsive REP system, (2.1.3) with* $k>0$ *and with nonzero background* $c_b>0$. *The solution of the* $3D$ *REP system remains bounded for all time if* $\lambda_{10}=\lambda_{20}=\lambda_{30}$.

Theorem 2.2.4. (Finite time blow up for 3D REP with k $>$ 0). *Consider the* $3D$ *repulsive REP system, (1.3) with* $k>0$ *and with nonzero background* $c_b>0$. *Assume* $\lambda(M_0)\in\mathbb{R}$. *The solution of the* $3D$ *REP system will blow up in finite time provided* $\rho_0>0$ *and one of the following three conditions are fulfilled:*

(i) $\lambda_{20}-\lambda_{10} > \left(\frac{k^3\rho_0^4}{4c_b}\right)^{\frac{1}{6}}$,

(ii) $\lambda_{20}-\lambda_{10} = \left(\frac{k^3\rho_0^4}{4c_b}\right)^{\frac{1}{6}}$ *and* $\lambda_{20}+\lambda_{10}<0$,

(iii) $0<\lambda_{20}-\lambda_{10} < \left(\frac{k^3\rho_0^4}{4c_b}\right)^{\frac{1}{6}}$ *and either* $\alpha>1$ *or* $\alpha\le 1$ *with*

$$\lambda_{20}+\lambda_{10} < sgn(1-\beta)\sqrt{(\beta+1)(\lambda_{20}-\lambda_{10})^2 - \frac{4k}{3}\left(2\rho_0 + c_b - \frac{3c_b}{\beta}\right)},$$

where α *and* β *with* $\alpha<\beta$ *are given by*

$$\frac{3}{4kc_b}(\lambda_{20}-\lambda_{10})^2 = -\frac{1}{\xi^2} + \frac{\rho_0}{c_b}\frac{1}{\sqrt{\xi}}, \quad \beta = \max\{\xi\}$$

and

$$\frac{3}{4kc_b}(\lambda_{20}-\lambda_{10})^2 = -\frac{1}{\alpha\beta} + \frac{2\rho_0}{c_b}\frac{1}{\sqrt{\alpha}+\sqrt{\beta}}.$$

Remark 2.2.5. *Some remarks are in order at this point.*

(i) In Theorem 2.2.1 and 2.2.2, the threshold bound denoted by $\Lambda_3(k,\rho_0)$ *is well defined for* $k<0$ *since the quantity under the square root is nonnegative. i.e.,*

$$k\left(c_b^{\frac{1}{3}}\rho_0^{\frac{2}{3}} - \frac{2}{3}\rho_0 - \frac{1}{3}c_b\right) = -\frac{k}{3}\left(2\rho_0^{\frac{1}{3}} + c_b^{\frac{1}{3}}\right)\left(\rho_0^{\frac{1}{3}} - c_b^{\frac{1}{3}}\right)^2 \ge 0.$$

(ii) From Theorem 2.2.1 and 2.2.2, we see that for each fixed ρ_0, *the lower bound in (2.2.1) for global existence and the upper bound in (2.2.2) for finite time blow up are identical. Thus,*

the obtained thresholds are optimal. In the sense that if $\lambda_{10} = \lambda_{30}$, then Theorem 2.2.1 and 2.2.2 can be combined into one theorem with an "if and only if" statement; otherwise, if the bound $\Lambda_3(k, \rho_0)$ lies between λ_{10} and λ_{30}, i.e.,

$$\lambda_{10} < \Lambda_3(k, \rho_0) \leq \lambda_{30},$$

it is unclear whether the C^1 solution regularity persists for all time.

(iii) The set of the initial configurations which give rise to global bounded solutions is very rich in phase space $(\rho, \lambda_1, \lambda_2, \lambda_3)$, which can be visualized through a qualitative diagram in the subspace $(\lambda_1, \lambda_3 - \lambda_1, \rho)$ (Figure 1). From the figure, one may also see that a critical threshold surface should lie somewhere between two shaded surfaces.

(iv) The condition for global regularity in Theorem 2.2.3 is obtained using only a global invariant, which is a set of measure zero in the space of eigenvalues with $\rho_0 > 0$. This global existence result, though starting from a thin initial set, when combined with Theorem 2.2.4 does suggest the existence of a critical threshold for the case $k > 0$. It would be interesting to identify a larger set of initial data than that in Theorem 2.2.3 for the global existence.

(v) For $k > 0$ case, the spectral gap $\lambda_2 - \lambda_1$ as described in Theorem 2.2.4 plays an important role. This fact is consistent with the known result in 2D case (Theorem 1.2 in [Liu et al. (2003)]).

(vi) The results in Theorems 2.2.1-2.2.4 may suggest the critical threshold phenomena for the full Euler-Poisson equations.

For the proof of each theorem, we need the following lemma.

Lemma 2.2.6. (Spectral dynamics [10, Lemma 3.1]). *Consider nonlinear transport equation $\mathbf{u}_t + \mathbf{u} \cdot \nabla_x \mathbf{u} = \vec{F}$. Let $\lambda := \lambda(\nabla_x \mathbf{u})(t, x)$ denote an eigenvalue of $\nabla_x \mathbf{u}$ with corresponding left and right normalized eigenpair, $\langle l, r\rangle = 1$. Then λ is governed by the forced Riccati equation*

$$\lambda' + \lambda^2 := \partial_t \lambda + \mathbf{u} \cdot \nabla_x \lambda + \lambda^2 = \langle l, \nabla_x \vec{F} r\rangle.$$

This lemma when applied to (2.1.3) gives

$$\lambda_i' + \lambda_i^2 = \frac{k}{n}(\rho - c_b), \quad i \in \{1, \cdots, n\}, \tag{2.2.3a}$$

$$\rho' + \rho(\lambda_1 + \cdots + \lambda_n) = 0. \tag{2.2.3b}$$

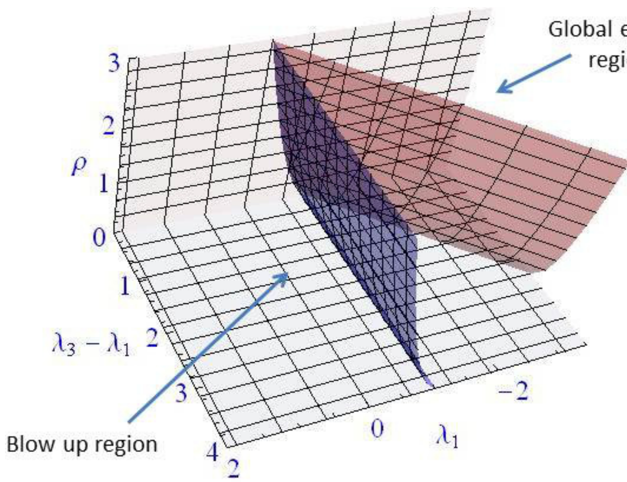

Figure 2.1 The sub and super-thresholds are shaded surfaces when $k < 0$

From (2.2.3a) it follows that

$$(\lambda_i - \lambda_j)' = -(\lambda_i - \lambda_j)(\lambda_i + \lambda_j), \quad i, j \in \{1, \cdots, n\}.$$

This shows that if $\lambda(M_0) \in \mathbb{R}$, then $\lambda(M) \in \mathbb{R}$. Moreover, if $\lambda(M_0) \in \mathbb{R}$, then the order of $\{\lambda_i\}_{i=1}^n$ is preserved in time, i.e.,

$$If\ \lambda_1(0) \leq \cdots \leq \lambda_n(0),\ then\ \lambda_1(t) \leq \cdots \leq \lambda_n(t)\ for\ t \geq 0. \tag{2.2.4}$$

This monotonicity preserving property remains valid in a strict sense because $\lambda_i - \lambda_j \neq 0$ as long as $\lambda_{i0} - \lambda_{j0} \neq 0$. Note that (2.2.3) is a (n+1)-by-(n+1) ODE system, when $n \geq 3$, it is no longer possible to employ the precise phase plane analysis as carried out in [Engelberg et al. (2001)] for (u_x, ρ) and in [Liu et al. (2003)] for (β, ρ), with β being a combined quantity of two eigenvalues through a global invariant. The key argument in our proofs here is to use the order preserving property of eigenvalues.

In the proof of Theorem 2.2.1-2.2.2 with $k < 0$, the order preserving property of $\lambda(M)$ together with non-negativity of the density enables us to obtain the following

$$-n\rho\lambda_n \leq \rho' \leq -n\rho\lambda_1.$$

This two sided differential inequality leads to the desired thresholds for both global existence and the finite time blow up. In the presence of complex eigenvalues, we also need the following order preserving property to prove global boundedness of the imaginary part of eigenvalues. If $\lambda_1(0)$ is real and

$$\lambda_1(0) \le \mathrm{Re}(\lambda_j(0)),$$

then

$$\lambda_1(t) \le \mathrm{Re}(\lambda_j(t))$$

for $t \ge 0,\ j \in \{2, \cdots, n\}$. In the proof of Theorem 2.2.4 with $k > 0$, we use the order preserving property to deduce some 2-by-2 ODE systems with controllable time-dependent coefficients, which when combined with some comparison argument leads to the desired blow up results.

From system (2.2.3), it follows that

$$(\lambda_i - \lambda_j) = (\lambda_{i0} - \lambda_{j0})e^{-\int_0^t (\lambda_i + \lambda_j)\, ds}, \quad i \in \{1, 2, 3\},$$

and

$$\rho = \rho_0 e^{-\int_0^t (\lambda_1 + \lambda_2 + \lambda_3)\, ds}.$$

These combined lead to the spectral invariant as obtained in [Liu et al. (2002)], i.e.,

$$S(t) := \frac{(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1)}{\rho^2} = S(0). \tag{2.2.5}$$

Our results are obtained from comparison of eigenvalues of the original system to solutions of dominated systems, which implicitly follow the order indicated by this spectral invariant, therefore our results are consistent with (2.2.5). This comment also applies to the higher-dimensional case.

We point out that for $k < 0$ case, only density and one eigenvalue need to be controlled for proving the global existence or the finite time blow up. Hence for $k < 0$ case, the key arguments summarized above work equally well for arbitrary n-dimensional REP equations ($n > 3$).

For $k > 0$ case, our argument for solution blow up extends only to 4-dimensional REP equations.

For completeness, we also state n-dimensional results for $k < 0$ case, 4-dimensional blow up result and n-dimensional global existence result for $k > 0$ case below and shall outline some main arguments of their proofs in §4.

Theorem 2.2.7. (Extention of Theorem 2.2.1). *Consider the nD attractive REP system, (1.3) with $k < 0$ and with nonzero background $c_b > 0$. Assume $\lambda_{10} \in \mathbb{R}$ and $\lambda_{10} \leq \mathrm{Re}(\lambda_{i0})$, $i = 2, 3, ...n$. The solution of the nD REP system remains bounded for all time if $\rho_0 > 0$ and*

$$\lambda_{10} > \mathrm{sgn}(\rho_0 - c_b)\sqrt{k\left(\frac{c_b^{\frac{n-2}{n}}}{n-2}\rho_0^{\frac{2}{n}} - \frac{2}{n(n-2)}\rho_0 - \frac{c_b}{n}\right)}.$$

Theorem 2.2.8. (Extention of Theorem 2.2.2). *Consider the nD attractive REP system, (1.3) with $k < 0$ and with nonzero background $c_b > 0$. Assume $\lambda(M_0) \in \mathbb{R}$. The solution of the nD REP system will blow up in finite time if $\rho_0 > 0$ and*

$$\lambda_{n0} < \mathrm{sgn}(\rho_0 - c_b)\sqrt{k\left(\frac{c_b^{\frac{n-2}{n}}}{n-2}\rho_0^{\frac{2}{n}} - \frac{2}{n(n-2)}\rho_0 - \frac{c_b}{n}\right)}. \tag{2.2.6}$$

Theorem 2.2.9. (Extention of Theorem 2.2.3). *Consider the nD repulsive REP system, (2.1.3) with $k > 0$ and with nonzero background $c_b > 0$. The solution of the nD REP system remains bounded for all time if all eigenvalues are initially real and identical.*

Theorem 2.2.10. (Finite time blow up for 4D REP with k > 0). *Consider the 4D repulsive REP system, (1.3) with $k > 0$ and with nonzero background $c_b > 0$. Assume $\lambda(M_0) \in \mathbb{R}$. The solution of the 4D REP system will blow up in finite time if $\rho_0 > 0$ and $\lambda_{20} - \lambda_{10} \geq \sqrt{k\rho_0}$.*

Remark 2.2.11. *Two remarks are in order at this point.*

i) The bound $\Lambda_n(k, \rho_0)$ in Theorem 2.2.7 and 2.2.8 is well defined since

$$\begin{cases} -\frac{k}{n}\frac{1}{(\frac{n}{2}-1)}\left(c_b^{\frac{2}{n}} - \rho_0^{\frac{2}{n}}\right)^2\left(\sum_{j=1}^{\frac{n}{2}-1} j c_b^{\frac{j-1}{(n/2)}} \rho_0^{\frac{(n/2)-1-j}{(n/2)}}\right) \geq 0, & n\text{-even}, \\ -\frac{k}{n}\frac{1}{(n-2)}\left(c_b^{\frac{1}{n}} - \rho_0^{\frac{1}{n}}\right)^2\left((n-2)c_b^{\frac{n-2}{n}} + \sum_{j=1}^{n-2} 2j c_b^{\frac{j-1}{n}} \rho_0^{\frac{n-1-j}{n}}\right) \geq 0, & n\text{-odd}. \end{cases}$$

ii) Under the assumptions of Theorem 2.2.8, we may use trace of M to derive a different threshold for the finite time blow up. In fact, taking trace of (2.1.3a) *we obtain*

$$(\mathrm{tr}(M))' + \mathrm{tr}(M^2) = k(\rho - c_b),$$

which holds for both full Euler-Poisson equations and Restricted Euler-Poisson equations. When $\lambda(M) \in \mathbb{R}$, we have

$$\mathrm{tr}(M^2) \geq \frac{1}{n}(\mathrm{tr}(M))^2. \tag{2.2.7}$$

Hence the trace $d = \mathrm{tr}(M) = \sum_{i=1}^{n} \lambda_i$ *satisfies*

$$d' \leq -\frac{d^2}{n} + k(\rho - c_b). \tag{2.2.8}$$

This when combined with $\rho' = -\rho d$ *leads to the following blow up condition:*

The solution of the nD REP system will blow up in finite time if $\rho_0 > 0$ and

$$\frac{d_0}{n} < \mathrm{sgn}(\rho_0 - c_b)\sqrt{k\left(\frac{c_b^{\frac{n-2}{n}}}{n-2}\rho_0^{\frac{2}{n}} - \frac{2}{n(n-2)}\rho_0 - \frac{c_b}{n}\right)}. \tag{2.2.9}$$

This threshold condition is slightly sharper than (2.2.6).

We note that the same threshold condition (2.2.9) for finite time blow up is obtained in [Cheng et al. (2009)] for the full EP system by assuming $\nabla \times \mathbf{u}_0 = \mathbf{0}$*, with which* M_0 *is symmetric, so is* $M(t)$ *for* $t > 0$*. This ensures that* $\lambda(M)$ *is real for all time. In contrast, for the REP system* $\lambda(M)$ *remains real as long as it is real at* $t = 0$*.*

2.3 Attractive Forcing Case, $k < 0$

We start this section with a lemma which compares the following two ODE systems,

$$\begin{cases} \rho' = \alpha\rho\lambda + \rho f(t), \\ \lambda' = \beta\lambda^2 + k\rho + \gamma, \end{cases} \tag{2.3.1}$$

and

$$\begin{cases} a' = \alpha ab, \\ b' = \beta b^2 + ka + \gamma. \end{cases} \tag{2.3.2}$$

Here, α, β, γ and k are fixed constants and $f(t)$ is a continuous function.

Proposition 2.3.1. *Let* $\alpha, k < 0$.

If $f(t) \geq 0$, $\forall t \geq 0$, *then*

$$\begin{cases} a(0) < \rho(0) \\ \lambda(0) < b(0) \end{cases} \quad \textit{implies} \begin{cases} a(t) < \rho(t), \\ \lambda(t) < b(t). \end{cases}$$

If $f(t) \leq 0$, $\forall t \geq 0$, *then*

$$\begin{cases} \rho(0) < a(0) \\ b(0) < \lambda(0) \end{cases} \quad \textit{implies} \begin{cases} \rho(t) < a(t), \\ b(t) < \lambda(t). \end{cases}$$

Proof. It can be proved by contradiction. Let $f(t) \geq 0$ and suppose t_1 is the earliest time when the above proposition is violated. Then

$$\begin{aligned} a(t_1) &= a(0)e^{\int_0^{t_1} \alpha b\, ds} \\ &< \rho(0)e^{\int_0^{t_1} \alpha\lambda\, ds} \\ &\leq \rho(0)e^{\int_0^{t_1} \alpha\lambda\, ds}e^{\int_0^{t_1} f(s)\, ds} \\ &= \rho(t_1). \end{aligned} \tag{2.3.3}$$

Therefore, it is left with only one possibility $\lambda(t_1) = y(t_1)$. Consider

$$(b-\lambda)' = \beta(b^2 - \lambda^2) + k(a-\rho). \tag{2.3.4}$$

Since $b(t) - \lambda(t) > 0$ for $t < t_1$ and $b(t_1) - \lambda(t_1) = 0$, hence at $t = t_1$, we have

$$(b(t_1) - \lambda(t_1))' \leq 0.$$

But the right hand side of (2.3.4) when it is evaluated at $t = t_1$,

$$k(a(t_1) - \rho(t_1)) > 0,$$

leads to the contradiction, as needed. The proof of $f(t) \leq 0$ case is similar. □

2.3.1 Proof of the global existence for 3D REP

As we remarked in the introduction, the spectral dynamics lemma tells us that the velocity gradient equation yields

$$\lambda_1' + \lambda_1^2 = \frac{k(\rho - c_b)}{3}, \tag{2.3.5a}$$

$$\lambda_2' + \lambda_2^2 = \frac{k(\rho - c_b)}{3}, \tag{2.3.5b}$$

$$\lambda_3' + \lambda_3^2 = \frac{k(\rho - c_b)}{3}, \tag{2.3.5c}$$

$$\rho' + \rho(\lambda_1 + \lambda_2 + \lambda_3) = 0. \tag{2.3.5d}$$

We first show the order preserving property of eigenvalues. As we showed in the introduction, if $\lambda(M_0) \in \mathbb{R}$, then $\lambda(M) \in \mathbb{R}$ and

$$\lambda_{10} \leq \lambda_{20} \leq \lambda_{30} \text{ implies } \lambda_1(t) \leq \lambda_2(t) \leq \lambda_3(t) \text{ for } t \geq 0.$$

Note that the gradient velocity matrix $M(t)$ is a real matrix, therefore, its eigenvalues are generically in complex conjugate pairs. In case $\lambda(M_0) \in \mathbb{C}$, the above property also holds in the following sense.

Lemma 2.3.2. *Assume $\lambda_{10} \in \mathbb{R}$. Then for $j \in \{2,3\}$,*

$$\mathrm{Re}(\lambda_{j0}) - \lambda_{10} \geq 0 \quad \textit{implies} \quad \mathrm{Re}(\lambda_j(t)) - \lambda_1(t) \geq 0,$$

as long as they remain finite.

Proof. Let $\lambda_j = \alpha + \beta i$. Then the real part of (2.3.5b) (or (2.3.5c)) leads to

$$\alpha' = -(\alpha^2 - \beta^2) + \frac{k(\rho - c_b)}{3}. \tag{2.3.6}$$

By subtracting (2.3.5a) from the above equation, we get

$$(\alpha - \lambda_1)' = -(\alpha^2 - \lambda_1^2) + \beta^2 \geq -(\alpha - \lambda_1)(\alpha + \lambda_1).$$

Thus, $(\alpha_0 - \lambda_{10}) \geq 0$ implies that $(\alpha - \lambda_1)(t) \geq 0$. □

In order to show the global existence, we rewrite (2.3.5a) and (2.3.5d) as

$$\begin{cases} \rho' = -\rho(\lambda_1 + \lambda_2 + \lambda_3) = -\rho(3\lambda_1) - \rho(\lambda_2 + \lambda_3 - 2\lambda_1), \\ \lambda_1' = -\lambda_1^2 + \frac{k(\rho - c_b)}{3}. \end{cases} \tag{2.3.7}$$

Comparing this with the following ODE

$$\begin{cases} a' = -3ab, \\ b' = -b^2 + \frac{k(a - c_b)}{3}, \end{cases} \tag{2.3.8}$$

we find the following monotonicity relation between (2.3.7) and (2.3.8).

Lemma 2.3.3. *Assume $k < 0$, $\lambda_{10} \in \mathbb{R}$ and $\lambda_{10} \leq \mathrm{Re}(\lambda_{20}) \leq \mathrm{Re}(\lambda_{30})$. Then*

$$\begin{cases} \rho(0) < a(0) \\ b(0) < \lambda_1(0) \end{cases} \quad \textit{implies} \quad \begin{cases} \rho(t) < a(t) \\ b(t) < \lambda_1(t) \end{cases} \quad \textit{for } t \geq 0,$$

as long as they remain finite.

Proof. The order preserving property in Lemma 4.3.3 gives $-\rho(\lambda_2 + \lambda_3 - 2\lambda_1) \leq 0$. Hence, by Proposition 2.3.1, the above lemma follows. □

Note that the modified ODE system (2.3.8) admits three critical points:

$$(0, b_\pm) := \left(0, \pm\sqrt{\frac{-kc_b}{3}}\right) \quad \text{and} \quad (c_b, 0).$$

One can verify that $(0, b_+)$ is a nodal sink, $(0, b_-)$ is a nodal source and $(c_b, 0)$ is a saddle point. We now use these facts to construct the threshold via phase plane analysis. Following the same q-transformation as that employed in [Liu et al. (2002)], we set $q = b^2$ to obtain

$$\frac{dq}{da} = 2b\frac{b'}{a'} = \frac{2q}{3a} - \frac{2k(a - c_b)}{9a},$$

which yields

$$\frac{d}{da}(a^{-\frac{2}{3}}q) = -\frac{2}{9}ka^{-\frac{2}{3}}(1 - c_b a^{-1}). \tag{2.3.9}$$

Upon integration, a global invariant of system (2.3.8) is given by

$$\frac{b^2 + \frac{2}{3}ka + \frac{1}{3}kc_b}{a^{\frac{2}{3}}} = const. \tag{2.3.10}$$

Therefore, the separatrix at $(c_b, 0)$ is given by zero level set

$$\frac{b^2 + \frac{2}{3}ka + \frac{1}{3}kc_b}{a^{\frac{2}{3}}} - kc_b^{\frac{1}{3}} = 0.$$

This gives the following lemma.

Lemma 2.3.4. *Consider system* (2.3.8)*, subject to initial data* (a_0, b_0)*. If* $(a_0, b_0) \in \Omega_1$*, then* $\lim_{t\to\infty}(a(t), b(t)) = (0, b_+) = \big(0, \sqrt{\frac{-kc_b}{3}}\big)$*. Here,*

$$\Omega_1 := \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; x > 0,\ y > sgn(x - c_b)\sqrt{k\left(c^{\frac{1}{3}b}x^{\frac{2}{3}} - \frac{2}{3}x - \frac{1}{3}c_b\right)}\right\}.$$

Proof. Note that Ω_1 is an invariant region for modified system (2.3.8) and $(0, b_+)$ is the nodal sink. From these facts, the lemma follows. □

Since Ω_1 is an open set and an invariant region of modified system (2.3.8), if $(\rho_0, \lambda_{10}) \in \Omega_1$ and

$$\lambda_{10} \le \mathrm{Re}(\lambda_{20}) \le \mathrm{Re}(\lambda_{30}),$$

then Lemma 2.3.3 and 2.3.4 gives the lower bound of λ_1, i.e.,

$$\lambda^* \le \lambda_1(t) \le \mathrm{Re}(\lambda_2(t)) \le \mathrm{Re}(\lambda_3(t)) \quad \text{where } \lambda^* = \min\left\{\lambda_1(0),\ \sqrt{\frac{-kc_b}{3}}\right\}. \tag{2.3.11}$$

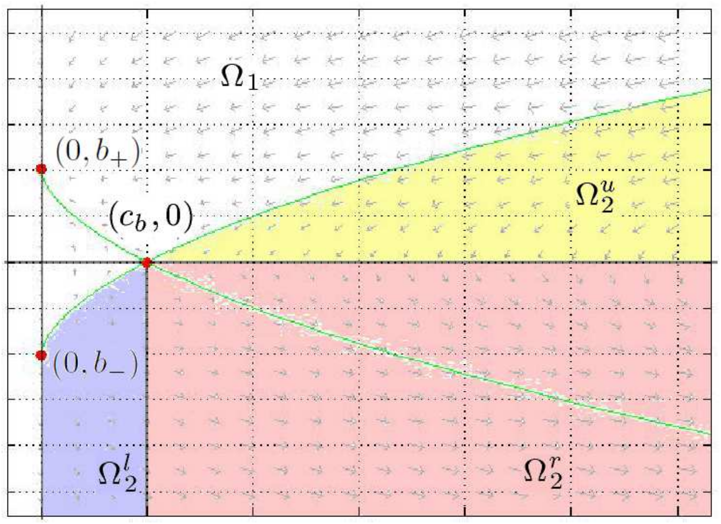

Figure 2.2 Ω_1 and Ω_2 for $k < 0$

If $\lambda(M_0) \in \mathbb{R}$, then it suffices to show that λ_3 is bounded from above. From (2.3.5c),

$$\lambda_3' = -\lambda_3^2 + \frac{k(\rho - c_b)}{3} \leq -\lambda_3^2 - \frac{kc_b}{3}, \tag{2.3.12}$$

we have

$$\lambda_3' < -\left(\lambda_3 + \sqrt{-\frac{kc_b}{3}}\right)\left(\lambda_3 - \sqrt{-\frac{kc_b}{3}}\right).$$

Thus,

$$\lambda_3(t) \leq \max\left\{\lambda_3(0),\ \sqrt{-\frac{kc_b}{3}}\right\}.$$

Together with (3.2.10), this proves Theorem 2.2.1 when $\mathrm{Im}(\lambda_{j0}) = 0$, $j \in \{2, 3\}$.

If $\mathrm{Im}(\lambda_{j0}) \neq 0$ for some $j \in \{2, 3\}$, then we need to bound both $\alpha(t) := \mathrm{Re}(\lambda_j(t))$ and $\beta(t) := \mathrm{Im}(\lambda_j(t))$. We show that there exists uniform upper bounds of $\alpha(t)$ and $|\beta(t)|$.

Lemma 2.3.5. *Assume $\lambda_{10} \in \mathbb{R}$ and $\mathrm{Im}(\lambda_{j0}) \neq 0$. If $(\rho_0, \lambda_{10}) \in \Omega_1$ and $\lambda_{10} \leq \mathrm{Re}(\lambda_{j0})$, then*

$$\alpha(t) \leq \max\left\{\mathrm{Re}(\lambda_{j0}),\ \sqrt{\left\{\mathrm{Im}(\lambda_{j0})K^*\right\}^2 - \frac{kc_b}{3}}\right\}$$

and

$$|\beta(t)| \leq |\mathrm{Im}(\lambda_{j0})|K^*,$$

where K^* is a constant independent of t.

Proof. From the imaginary part of (2.3.5b), we have $\beta' = -2\alpha\beta$, hence

$$|\beta(t)| = \left|\beta(0)e^{-\int_0^t 2\alpha(s)\,ds}\right| \le |\beta(0)|e^{-\int_0^t 2\lambda_1(s)\,ds}, \tag{2.3.13}$$

where the inequality comes from Lemma 4.3.3. Note that Ω_1 is an open set and given any initial data $(\rho_0, \lambda_{10}) \in \Omega_1$ for system (2.3.7), we can find $\epsilon > 0$ and initial data $(a(0), b(0)) := (\rho_0 + \epsilon, \lambda_{10} - \epsilon) \in \Omega_1$ for modified system (2.3.8). Therefore, by Lemma 2.3.3 and the fact that there exists time $T^* \ge 0$ such that $b(t) > 0$ for all $t \ge T^*$, we have

$$e^{-\int_0^t 2\lambda_1(s)\,ds} \le e^{-\int_0^t 2b(s)\,ds} \le \max_{0 \le t \le T^*} \left\{e^{-\int_0^t 2b(s)\,ds}\right\} =: K^*.$$

This gives, $|\beta(t)| \le |\beta(0)|K^*$. Also, by (2.3.5a) and the upper bound of $|\beta(t)|$, we have

$$\begin{aligned} \alpha'(t) &< -\alpha^2(t) + (|\beta(0)|K^*)^2 - \frac{kc_b}{3} \\ &= -\left(\alpha(t) + \sqrt{(|\beta(0)|K^*)^2 - \frac{kc_b}{3}}\right)\left(\alpha(t) - \sqrt{(|\beta(0)|K^*)^2 - \frac{kc_b}{3}}\right). \end{aligned} \tag{2.3.14}$$

Thus,

$$\alpha(t) \le \max\left\{\alpha(0),\ \sqrt{(|\beta(0)|K^*)^2 - \frac{kc_b}{3}}\right\}.$$

□

Together with (3.2.10), this completes the proof of Theorem 2.2.1.

2.3.2 Proof of the finite-time blow up for 3D REP

For the blow up condition, we rewrite (2.3.5c) and (2.3.5d)

$$\begin{cases} \rho' = -\rho(\lambda_1 + \lambda_2 + \lambda_3) = -\rho(3\lambda_3) - \rho(\lambda_1 - \lambda_3) - \rho(\lambda_2 - \lambda_3), \\ \lambda_3' = -\lambda_3^2 + \frac{k(\rho - c_b)}{3}. \end{cases} \tag{2.3.15}$$

Similarly, we shall compare the above system with the following modified system:

$$\begin{cases} a' = -3ab, \\ b' = -b^2 + \frac{k(a - c_b)}{3}. \end{cases} \tag{2.3.16}$$

Following a similar proof to that of Lemma 2.3.3, we find the monotonicity relation between (2.3.15) and (2.3.16).

Lemma 2.3.6. *Assume $\lambda(M_0) \in \mathbb{R}$ and $\lambda_{10} \leq \lambda_{20} \leq \lambda_{30}$. Then*

$$\begin{cases} a(0) < \rho(0) \\ \lambda_3(0) < b(0) \end{cases} \quad \textit{implies} \quad \begin{cases} a(t) < \rho(t) \\ \lambda_3(t) < b(t) \end{cases} \quad \textit{for } t \geq 0.$$

We shall prove the blow up of solutions to modified system (2.3.16), i.e., $b(t) \to -\infty$ in finite time, which in turn, by Lemma 2.3.6 implies $\lambda_3(t) \to -\infty$ in finite time.

Note that system (2.3.16) is the same as (2.3.8). We thus have the same global invariant as (2.3.10). Hence, from the separatrix curve given by

$$\frac{b^2 + \frac{2}{3}ka + \frac{1}{3}kc_b}{a^{\frac{2}{3}}} - kc_b^{\frac{1}{3}} = 0,$$

we can show the blow up region of system (2.3.16).

Lemma 2.3.7. *Consider the modified system* (2.3.16)*, subject to initial data* (a_0, b_0)*. If* $(a_0, b_0) \in \Omega_2$*, then* $b \to -\infty$*,* $a \to \infty$ *at a finite time. Here,*

$$\Omega_2 := \left\{ (x, y) \;\middle|\; x > 0,\; y < \mathrm{sgn}(x - c_b)\sqrt{k\left(c^{\frac{1}{3}}{}_b x^{\frac{2}{3}} - \frac{2}{3}x - \frac{1}{3}c_b\right)} \right\}.$$

Proof. Note that Ω_2 is an invariant region, which is decomposed as $\Omega_2^l \cap \Omega_2^r \cap \Omega_2^u$ with

$$\Omega_2^l := \Omega_2 \cap \{(x, y) \mid x \leq c_b\}, \quad \Omega_2^r := \Omega_2 \cap \{(x, y) \mid x > c_b,\; y < 0\}$$

and $\Omega_2^u := \Omega_2 \cap \{(x, y) \mid x > c_b,\; y \geq 0\}$ (see Figure 2 in Section 3.1). It is straightforward to verify that if $(a_0, b_0) \in \Omega_2^l \cup \Omega_2^u$, then $(a(t), b(t)) \in \Omega_2^r$ in finite time. Note that if $(a_0, b_0) \in \Omega_2^r$, then $a(t)$ is increasing in t. Thus, $a(t) > c_b$, $\forall t$. This implies $b' < -b^2$, which upon integration yields

$$b(t) < \frac{b_0}{tb_0 + 1}.$$

Hence, the blow up time t^B of $b(t)$ must satisfy

$$t^B < -\frac{1}{b_0}.$$

Also $a(t)$ approaches ∞ in finite time due to the global invariant (2.3.10). $\square$

The last step of proving Theorem 2.2.2 is to combine the comparison principle in Lemma 2.3.6 with Lemma 2.3.7. We notice that Ω_2 is an open set and for any given initial data

$(\rho_0, \lambda_{30}) \in \Omega_2$ for original system (2.3.15), we can always find $\epsilon > 0$ such that initial data $(\rho_0 - \epsilon, \lambda_{30} + \epsilon) \in \Omega_2$ for modified system (2.3.16). This latter initial data will lead to finite time blow up of the modified system and thus the initial data $(\rho_0, \lambda_{30}) \in \Omega_2$ will lead to finite time blow up of the original system.

2.4 Repulsive Forcing Case, $k > 0$

2.4.1 Proof of the global existence for 3D REP

This subsection is devoted to the proof of global existence for REP with $k > 0$. The spectral dynamics lemma tells us that the velocity gradient equation yields

$$\begin{cases} \lambda_i' = -\lambda_i^2 + \frac{k(\rho - c_b)}{3}, \qquad i = 1, 2, 3, \\ \rho' = -\rho(\lambda_1 + \lambda_2 + \lambda_3). \end{cases} \tag{2.4.1}$$

Since $\lambda_{10} = \lambda_{20} = \lambda_{30}$, by the first equation of (2.4.1), we have $\lambda_1(t) = \lambda_2(t) = \lambda_3(t)$, $\forall$ $t \geq 0$. Let $\lambda := \lambda_i$, then by (2.4.1) we have,

$$\begin{cases} \lambda' = -\lambda^2 + \frac{k(\rho - c_b)}{3}, \\ \rho' = -3\rho\lambda. \end{cases} \tag{2.4.2}$$

To obtain a global invariant we set $q := \lambda^2$; then from (2.4.2) we deduce

$$\frac{dq}{d\rho} = 2\lambda \frac{\lambda'}{\rho'} = -\frac{2}{3\rho}\left(-q + \frac{k(\rho - c_b)}{3} \right).$$

Against the integrating factor of $\rho^{-\frac{2}{3}}$, we have

$$\frac{d}{d\rho}\left(\rho^{-\frac{2}{3}} q \right) = -\frac{2}{9} k \rho^{-\frac{2}{3}} + \frac{2kc_b}{9} \rho^{-\frac{5}{3}}.$$

Integrations with $q = \lambda^2$ gives,

$$\rho^{-\frac{2}{3}} \lambda^2 = -\frac{2k}{3} \rho^{\frac{1}{3}} - \frac{kc_b}{3} \rho^{-\frac{2}{3}} + Const$$

or

$$\frac{\lambda^2 + \frac{2k}{3}\rho + \frac{kc_b}{3}}{\rho^{2/3}} = Const.$$

From which it follows that ρ is bounded from above and away from zero, which in turn gives the boundedness of λ for all $t \geq 0$. This complete the proof of Theorem 2.2.3.

2.4.2 Proof of the finite-time blow up for 3D REP

This section is devoted to the proof of finite time blow up for REP with $k > 0$. From (2.4.1), it follows that

$$\begin{aligned}(\lambda_2-\lambda_1)' &= -(\lambda_2-\lambda_1)(\lambda_2+\lambda_1),\\ (\lambda_2+\lambda_1)' &= -\lambda_1^2-\lambda_2^2-\frac{2kc_b}{3}+\frac{2k\rho}{3}.\end{aligned} \tag{2.4.3}$$

Let $x := \lambda_2 - \lambda_1$, $y := \lambda_2 + \lambda_1$ and

$$g(t) := \frac{2k}{3}\rho x^{-\frac{3}{2}}.$$

Then (2.4.3) becomes

$$x' = -xy, \tag{2.4.4a}$$

$$y' = -\frac{y^2}{2} + G(x, g(t)), \tag{2.4.4b}$$

where we have used the following

$$G(x, g(t)) := -\frac{x^2}{2} - \frac{2kc_b}{3} + g(t)x^{\frac{3}{2}}.$$

From (2.4.4a), we have

$$x(t) = x(0)e^{-\int_0^t y(s)\,ds},$$

hence $x(t) \equiv 0$ is an invariant. We thus consider only $x(0) = \lambda_{20} - \lambda_{10} > 0$ case. A simple calculation gives

$$\begin{aligned} g'(t) &= \left(\frac{2k}{3}\rho x^{-\frac{3}{2}}\right)'\\ &= \frac{2k}{3}x^{-\frac{3}{2}}\left(\rho' - \frac{3}{2}x^{-1}x'\rho\right)\\ &= \frac{2k}{3}\rho x^{-\frac{3}{2}}\left\{-(\lambda_1+\lambda_2+\lambda_3)+\frac{3}{2}y\right\}\\ &= \frac{2k}{3}\rho x^{-\frac{3}{2}}\left\{\frac{1}{2}(\lambda_1+\lambda_2)-\lambda_3\right\}\\ &\le 0.\end{aligned}$$

Here, the last inequality comes from the order preserving property of $\lambda(M)$ and $x(t) > 0$, $\forall t \ge 0$. Therefore $g(t)$ is non-increasing in time. This fact gives the bound of $g(t)$,

$$0 < g(t) \le g(0) = \frac{2k}{3}\frac{\rho_0}{x_0^{3/2}}.$$

Using the upper bound of $g(t)$, we arrive at the following observation:

Lemma 2.4.1. *The solution of* (2.4.4) *will blow up in finite time if one of the following two conditions are fulfilled:*
(i) $x_0 > \left(\frac{k^3\rho_0^4}{4c_b}\right)^{\frac{1}{6}}$,
(ii) $x_0 = \left(\frac{k^3\rho_0^4}{4c_b}\right)^{\frac{1}{6}}, \quad y_0 < 0.$

Proof. Since $0 < g(t) \le g(0)$, $\forall t \ge 0$, we have

$$G(x, g(t)) \le G(x, g(0)), \quad \forall\, x > 0.$$

Also, a simple calculation gives,

$$\max_{x>0}\{G(x, g(0))\} = \frac{1}{6}\frac{k^4\rho_0^4}{x_0^6} - \frac{2kc_b}{3}.$$

Therefore, from (2.4.4b) it follows that

$$y' \le -\frac{y^2}{2} + \frac{1}{6}\frac{k^4\rho_0^4}{x_0^6} - \frac{2kc_b}{3},$$

which gives the desired results. $\square$

Using the given initial data x_0 and ρ_0 we replace time dependant coefficient $g(t)$ of (2.4.4b) by

$$N := g(0) = \frac{2k}{3}\frac{\rho_0}{x_0^{\frac{3}{2}}}$$

and construct a corresponding new system. That is, finding the other blow up region of system (2.4.4) is carried out by comparison with the following system

$$\begin{cases} a' = -ab, \\ b' = -\frac{b^2}{2} + G(a, N). \end{cases} \tag{2.4.5}$$

From now on, we assume that

$$x_0 < \left(\frac{k^3\rho_0^4}{4c_b}\right)^{\frac{1}{6}},$$

so that the system (2.4.5) has two equilibrium points $(a_i^*, 0)$, $i = 1, 2$ with

$$0 < a_2^* < \frac{9N^2}{4} < a_1^*.$$

Indeed, $G(a,N)$ has its local maximum at $a = \frac{9}{4}N^2$ and $G(\frac{9}{4}N^2, N) > 0$. Further calculation showss that $(a_1^*, 0)$ is a saddle point and $(a_2^*, 0)$ is a spiral of ODE system (2.4.5).

We first show the monotonicity relation between (2.4.4) and (2.4.5).

Lemma 2.4.2.

$$\begin{cases} 0 < a_0 < x_0 \\ y_0 < b_0, \end{cases} \quad \textit{implies} \begin{cases} a(t) < x(t) \\ y(t) < b(t), \end{cases} \quad \forall t \geq 0,$$

as long as $a(t) > \frac{9}{4}N^2$, $\forall t \geq 0$.

Proof. It can be proved by contradiction. Suppose t_1 is the earliest time when the above proposition is violated, then

$$a(t_1) = a_0 e^{-\int_0^{t_1} b(s)\,ds} < x_0 e^{-\int_0^{t_1} b(s)\,ds} < x_0 e^{-\int_0^{t_1} y(s)\,ds} = x(t_1).$$

Therefore, it is left with only one possibility $y(t_1) - b(t_1) = 0$. From (2.4.4b) and the second equation of (2.4.5),

$$(b-y)' = -\frac{1}{2}(b-y)(b+y) + G(a,N) - G(x, g(t)). \tag{2.4.6}$$

At $t = t_1$, we have

$$(b-y)'(t_1) \leq 0.$$

But the right hand side of (2.4.6) is positive. In fact, when it is evaluated at $t = t_1$,

$$\begin{aligned} RHS &= G(a(t_1), N) - G(x(t_1), g(t_1)) \\ &\geq G(a(t_1), g(t_1)) - G(x(t_1), g(t_1)) \\ &= G_x(\eta, g(t_1))(a(t_1) - x(t_1)). \end{aligned}$$

The last equality comes from the mean value theorem with $\eta \in (a(t_1), x(t_1))$. Also,

$$\begin{aligned} G_x(\eta, g(t_1)) &= \eta^{\frac{1}{2}} \left(\frac{3}{2} g(t_1) - \eta^{\frac{1}{2}} \right) \\ &\leq \eta^{\frac{1}{2}} \left(\frac{3}{2} N - \eta^{\frac{1}{2}} \right) \\ &< 0, \end{aligned}$$

since $\eta \geq a(t_1) > \frac{9}{4}N^2$. Therefore, the right hand side of (2.4.6) is positive. This leads to a contradiction, as needed. $\square$

Now we want to find finite time blow up conditions for system (2.4.5), which in turn, by Lemma 2.4.2, implies the finite time blow up of the original system (2.4.4). To this end, we set $q := b^2$, then from (2.4.5), we deduce

$$\frac{dq}{da} = 2b \cdot \frac{b'}{a'} = \frac{q}{a} + a + \frac{4kc_b}{3a} - 2N\sqrt{a}.$$

So,

$$\frac{d}{da}\left(\frac{q}{a}\right) = 1 + \frac{4kc_b}{3a^2} - \frac{2N}{\sqrt{a}} = -\frac{2}{a^2}G(a, N).$$

Integration leads to a global invariant

$$\frac{b^2}{a} = -2\int_c^a \frac{G(a,N)}{a^2}\, da, \quad \text{where } c \text{ is some constant.} \tag{2.4.7}$$

By setting $(a, b) = (a_1^*, 0)$, we find the separatrix curve passing $(a_1^*, 0)$,

$$\frac{b^2}{a} = -2\int_{a_1^*}^a \frac{G(s,N)}{s^2}\, ds. \tag{2.4.8}$$

The above curve has two x intercepts. One is $(a_1^*, 0)$ and the other is denoted by $(a^*, 0)$ with $0 < a^* < a_2^*$. In fact, consider

$$\int_a^{a_1^*} \frac{G(s,N)}{s^2}\, ds = \int_a^{a_2^*} \frac{G(s,N)}{s^2}\, ds + \int_{a_2^*}^{a_1^*} \frac{G(s,N)}{s^2}\, ds.$$

Note that $G(a, N) \geq 0$, $\forall a \in [a_2^*, a_1^*]$ and $\lim_{a \to 0+} \int_a^{a_2^*} \frac{G(s,N)}{s^2}\, ds \to -\infty$. This proves the existence of intercept $(a^*, 0)$ and the following identity,

$$\int_{a^*}^{a_1^*} \frac{G(s,N)}{s^2}\, ds = 0. \tag{2.4.9}$$

Together with the comparison lemma, (2.4.8) gives the following results.

Lemma 2.4.3. *The solution of* (2.4.4) *will blow up in finite time if*

$$(x_0, y_0) \in \Omega_1,$$

where

$$\Omega_1 := \left\{(x, y) \mid x > \frac{9}{4}N^2 \text{ and } y < sgn(x - a_1^*)\sqrt{2x\int_x^{a_1^*} \frac{G(s,N)}{s^2}\, ds}\right\}.$$

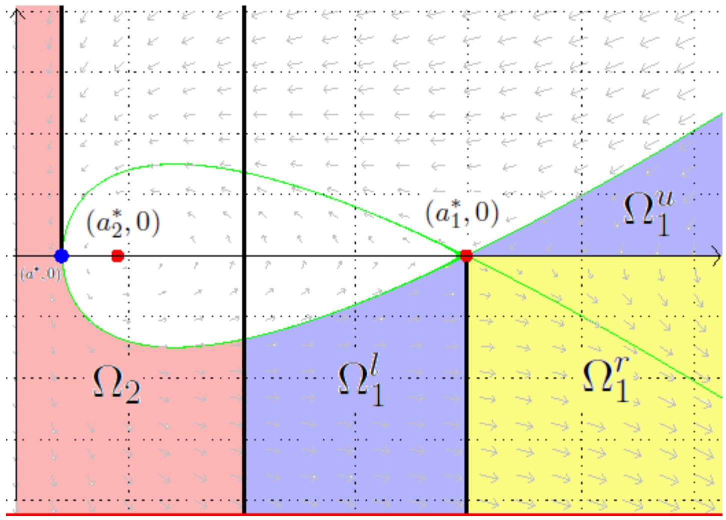

Figure 2.3 Ω_1 and Ω_2 for $k > 0$

Proof. Since we have the comparison between two systems (2.4.4) and (2.4.5), it suffices to show that the finite time blow up of solution for modified system (2.4.5). From (2.4.8), we know that Ω_1 is an invariant region, which is decomposed as $\Omega_1^l \cap \Omega_1^r \cap \Omega_1^u$ with

$$\Omega_1^l := \Omega_1 \cap \{(x,y)|\ x \leq \frac{9}{4}N^2\}, \quad \Omega_1^r := \Omega_1 \cap \{(x,y)|\ x > \frac{9}{4}N^2,\ y < 0\}$$

and $\Omega_1^u := \Omega_1 \cap \{(x,y)|\ y \geq 0\}$ (see Figure 3). It is straightforward to verify that if $(a_0, b_0) \in \Omega_1^l \cup \Omega_1^u$, then $(a(t), b(t)) \in \Omega_1^r$ in finite time. Therefore, without loss of generality, we let $(a_0, b_0) \in \Omega_1^r$, then $a(t) > a_1^*$ and $b(t) < 0$ for all $t \geq 0$. This implies

$$b' = -\frac{b^2}{2} + G(a, N) < -\frac{b^2}{2},$$

which upon integration yields

$$b(t) < \frac{2b_0}{tb_0 + 2}.$$

Hence, the blow up time t^B of $b(t)$ must satisfy

$$t^B < -\frac{2}{b_0}.$$

Also, $a(t)$ approaches ∞ in finite time due to the global invariant in (2.4.7). $\square$

The blow up condition in the above lemma was obtained by comparison with system (2.4.5) as long as $a(t) > \frac{9}{4}N^2$. In the region where $a(t) \leq \frac{9}{4}N^2$, we obtain blow up results by a different argument.

Lemma 2.4.4. *The solution of* (2.4.4) *will blow up in finite time if*

$$(x_0, y_0) \in \Omega_2^l \cup \Omega_2^r,$$

where

$$\Omega_2^l := \{(x,y)|\ 0 < x < a^*,\ \forall y\} \cup \{(x,y)|\ x = a^*,\ y \neq 0\}$$

and

$$\Omega_2^r := \left\{(x,y) \in \mathbb{R}^2|\ a^* < x \leq \frac{9}{4}N^2 \text{ and } y < -\sqrt{2x \int_x^{a_1^*} \frac{G(s,N)}{s^2}\, ds}\right\}.$$

Proof. In Lemma 2.4.1, we showed that $G(x, g(t)) \leq G(x, N)$, $\forall x > 0$. This gives the following ODI.

$$\begin{cases} x' = -xy, \\ y' \leq -\frac{y^2}{2} + G(x,N). \end{cases} \tag{2.4.10}$$

If $(x_0, y_0) \in \Omega_2^l$ with $y_0 \geq 0$, then from $x' = -xy$, we have $x(t) < a^*,\ \forall t > 0$ as long as $y \geq 0$. Hence

$$y' \leq G(x,N) \leq G(a^*, N) < 0.$$

Therefore, $y(t)$ will be negative after $t^* = -\frac{y_0}{G(a^*,N)}$.

We now consider $(x_0, y_0) \in \Omega_2^l$ with $y_0 < 0$; if $x(t) \leq a^*$ for all $t > 0$, we have

$$y' \leq -\frac{y^2}{2} + G(x,N) < -\frac{y^2}{2}.$$

This leads to the finite time blow up, unless $(x(t), y(t))$ enters Ω^r in finite time.

In such a case with $(x_0, y_0) \in \Omega_2^r$, we deduce

$$\frac{dy^2}{dx} = 2y \cdot \frac{y'}{x'} = -\frac{2}{x}y' \geq -\frac{2}{x}\Big(-\frac{y^2}{2} + G(x,N)\Big).$$

Therefore,

$$\frac{d}{dx}\left(\frac{y^2}{x}\right) \geq -\frac{2}{x^2}G(x,N).$$

Integration gives,

$$\frac{y^2}{x} - \frac{y_0^2}{x_0} \geq -2\int_{x_0}^{x} \frac{G(s,N)}{s^2}\,ds. \tag{2.4.11}$$

Now, consider a point (x_0, y_*) on separatrix curve (2.4.8) which is above (x_0, y_0). i.e.,

$$\frac{y_*^2}{x_0} = -2\int_{a_1^*}^{x_0} \frac{G(s,N)}{s^2}\,ds. \tag{2.4.12}$$

Since $y_0^2 > y_*^2$ and (2.4.11), we obtain

$$\frac{y^2}{x} + 2\int_{x_0}^{x} \frac{G(s,N)}{s^2}\,ds \geq \frac{y_0^2}{x_0} > \frac{y_*^2}{x_0} = -2\int_{a_1^*}^{x_0} \frac{G(s,N)}{s^2}\,ds.$$

We thus have

$$\frac{y^2}{x} > -2\int_{a_1^*}^{x} \frac{G(s,N)}{s^2}\,ds. \tag{2.4.13}$$

This relation shows that if $(x_0, y_0) \in \Omega_2^r$, then no $(x(t), y(t))$ crosses separatrix curve (2.4.8). Therefore, if $(x_0, y_0) \in \Omega_2^r$ with $x_0 \leq a_2^*$, then

$$(x(t), y(t)) \in \Omega^r \cap \{(x,y)|\ x > a_2^*\}$$

in finite time.

It is left to consider $(x_0, y_0) \in \Omega_2^r \cap \{(x,y)|\ x > a_2^*\}$. This set ensures that, $\exists \delta > 0$ such that

$$y_0^2 = \delta + 2x_0\int_{x_0}^{a_1^*} \frac{G(s,N)}{s^2}\,ds.$$

Therefore, from (2.4.11),

$$\begin{aligned}
\frac{y^2}{x} &\geq \frac{y_0^2}{x_0} - 2\int_{x_0}^{x} \frac{G(s,N)}{s^2}\,ds \\
&> \frac{\delta}{2x_0} + 2\int_{x_0}^{a_1^*} \frac{G(s,N)}{s^2}\,ds - 2\int_{x_0}^{x} \frac{G(s,N)}{s^2}\,ds \\
&\geq \frac{\delta}{2x_0} + 2\int_{x}^{a_1^*} \frac{G(s,N)}{s^2}\,ds \\
&\geq \frac{\delta}{2x_0},
\end{aligned} \tag{2.4.14}$$

where the last inequality comes from the fact that $a_2^* < x_0 < x(t)$, $\forall t > 0$,

$$G(x,N) \geq 0, x \in [a_2^*, a_1^*] \ \ and \ \ G(x,N) \leq 0, x \in [a_1^*, \infty].$$

By substituting the inequality in (2.4.14) into the first equation in (2.4.10), we obtain

$$\begin{aligned} x' &= -xy \\ &\geq x^{\frac{3}{2}}\sqrt{\frac{\delta}{2x_0}}, \quad \delta > 0. \end{aligned} \tag{2.4.15}$$

Therefore, $x(t) \to \infty$ in finite time. This gives the desired result. $\square$

By combining the blow up conditions in Lemma 3.3 and 3.4, we can get the following blow up condition: Either

$$0 < x_0 < a^* \tag{2.4.16}$$

or

$$x_0 \geq a^* \ \textit{with} \ y_0 < sgn(x_0 - a_1^*)\sqrt{x_0^2 + \left(a_1^* + \frac{4kc_b}{a_1^*}\right)x_0 - 4Nx_0^{\frac{3}{2}} - \frac{4kc_b}{3}}. \tag{2.4.17}$$

The last step of proving Theorem 2.2.4 is to convert the blow up conditions in (2.4.16) and (2.4.17) into conditions which involve the original variables ρ_0 and λ_i. Let $\beta := \frac{a_1^*}{x_0}$. Since

$$G(x, N) = -\frac{x^2}{2} - \frac{2kc_b}{3} + Nx^{\frac{3}{2}}$$

and $G(a_i^*, N) = 0$, $i = 1, 2$, we have

$$-\frac{(\beta x_0)^2}{2} - \frac{2kc_b}{3} + \frac{2k\rho_0}{3x_0^{3/2}}(\beta x_0)^{\frac{3}{2}} = 0.$$

This is equivalent to

$$\frac{3}{4kc_b}x_0^2 = -\frac{1}{\beta^2} + \frac{\rho_0}{c_b} \cdot \frac{1}{\sqrt{\beta}}. \tag{2.4.18}$$

Also, let $\alpha := \frac{a^*}{x_0}$. Since (2.4.8) pass through $(a^*, 0)$, we have

$$\begin{aligned} 0 &= \left(a^* - \frac{4kc_b}{3a^*} - 4N\sqrt{a^*}\right) - \left(a_1^* - \frac{4kc_b}{3a_1^*} - 4N\sqrt{a_1^*}\right) \\ &= \left(\alpha x_0 - \frac{4kc_b}{3\alpha x_0} - \frac{8k\rho_0}{3x_0^{3/2}}\sqrt{\alpha x_0}\right) - \left(\beta x_0 - \frac{4kc_b}{3\beta x_0} - \frac{8k\rho_0}{3x_0^{3/2}}\sqrt{\beta x_0}\right), \end{aligned} \tag{2.4.19}$$

or

$$(\alpha - \beta)x_0^2 - \frac{4kc_b}{3}\left(\frac{1}{\alpha} - \frac{1}{\beta}\right) - \frac{8k\rho_0}{3}(\sqrt{\alpha} - \sqrt{\beta}) = 0.$$

This is equivalent to

$$\frac{3}{4kc_b}x_0^2 = -\frac{1}{\alpha\beta} + \frac{2\rho_0}{c_b}\frac{1}{\sqrt{\alpha} + \sqrt{\beta}}. \tag{2.4.20}$$

With α, β introduced above, the blow up conditions in (2.4.16) and (2.4.17) can be written as

$$\alpha > 1$$

and

$$\alpha \leq 1 \ with \ \lambda_{20} + \lambda_{10} < sgn(1-\beta)\sqrt{(\beta+1)(\lambda_{20}-\lambda_{10})^2 - \frac{4k}{3}(2\rho_0 + c_b - \frac{3c_b}{\beta})},$$

respectively. This, together with Lemma 2.4.1, completes the proof of Theorem 2.2.4.

2.5 Extension to n-dimensions

In this section, we outline the proofs of the n-dimensional theorems. We also prove the 4-dimensional theorem for $k > 0$ case.

Proof of Theorem 2.2.7

From (1.5b) it follows that

$$\begin{aligned} \rho' &= -\rho(\sum_{i=1}^{n} \lambda_i) \\ &= -n\rho\lambda_1 - \rho\big\{\sum_{i=2}^{n} \lambda_i - (n-1)\lambda_1\big\} \\ &= -n\rho\lambda_1 - \rho\big\{\sum_{i=2}^{n} \mathrm{Re}(\lambda_i) - (n-1)\lambda_1\big\}. \end{aligned} \tag{2.5.1}$$

Consider any λ_j, $j \in \{2, ..., n\}$. If $\lambda_j \in \mathbb{R}$, then the order preserving property of real eigenvalues gives $\lambda_j - \lambda_1 \geq 0$. If $\mathrm{Im}(\lambda_j) \neq 0$, then Lemma 2.2 implies that $\mathrm{Re}(\lambda_j) - \lambda_1 \geq 0$. Thus,

$$-\rho\big\{\sum_{i=2}^{n} \mathrm{Re}(\lambda_i) - (n-1)\lambda_1\big\} \leq 0.$$

Therefore, ODE system

$$\begin{cases} \rho' = -n\rho\lambda_1 - \rho\big\{\sum_{i=2}^{n} \mathrm{Re}(\lambda_i) - (n-1)\lambda_1\big\}, \\ \lambda_1' = -\lambda_1^2 + \frac{k(\rho - c_b)}{n}, \end{cases}$$

can be compared with

$$\begin{cases} a' = -nab, \\ b' = -b^2 + \frac{k(a-c_b)}{n}. \end{cases} \tag{2.5.2}$$

This gives

$$\frac{db^2}{da} = \frac{2b^2}{na} - \frac{2k}{n^2} + \frac{2kc_b}{n^2 a},$$

that is,

$$\frac{d}{da}(a^{-\frac{2}{n}}b^2) = -\frac{2k}{n^2}a^{-\frac{2}{n}} + \frac{2kc_b}{n^2}a^{-1-\frac{2}{n}}.$$

Upon integration, the separatrix passing $(c_b, 0)$ is obtained and expressed by

$$b^2 = k\left(\frac{c_b^{\frac{n-2}{n}} a^{\frac{2}{n}}}{n-2} - \frac{2a}{n(n-2)} - \frac{c_b}{n}\right). \tag{2.5.3}$$

Using (2.5.3), define an invariant region of (2.5.2) by

$$\Omega_1' = \left\{(x, y) \mid x > 0,\ y > sgn(x - c_b)\sqrt{k\left(\frac{c_b^{\frac{n-2}{n}} x^{\frac{2}{n}}}{n-2} - \frac{2x}{n(n-2)} - \frac{c_b}{n}\right)}\right\}.$$

Since Ω_1' is an open set and an invariant region of system (2.5.2), for any given $(\rho_0, \lambda_{10}) \in \Omega_1'$, we can always find $\epsilon > 0$ and initial data

$$(a(0), b(0)) := (\rho_0 + \epsilon, \lambda_{10} - \epsilon) \in \Omega_1'$$

for system (2.5.2). Therefore, Proposition 2.3.1 gives

$$\lambda^* \le \lambda_1(t) \le \mathrm{Re}(\lambda_2(t)) \le \cdots \le \mathrm{Re}(\lambda_n(t)),\ \text{where } \lambda^* := \min\left\{\lambda_1(0),\ \sqrt{\frac{-kc_b}{n}}\right\}.$$

We now turn to find an upper bound of $\max_i\{\mathrm{Re}(\lambda_i(t))\}$ and $\max_i\{|\mathrm{Im}(\lambda_i(t))|\}$. For any $j \in \{1, ..., n\}$, let $\alpha = \mathrm{Re}(\lambda_j)$ and $\beta = \mathrm{Im}(\lambda_j)$, then by (2.5),

$$\alpha' = -\alpha^2 + \beta^2 + \frac{k}{n}(\rho - c_b).$$

If $\beta(0) = 0$, then $\beta(t) = 0$ and

$$\alpha' \le -\alpha^2 - \frac{kc_b}{n} = -\left(\alpha + \sqrt{-\frac{kc_b}{n}}\right)\left(\alpha - \sqrt{-\frac{kc_b}{n}}\right).$$

This gives $\lambda_j(t) \le \max\left\{\lambda_{j0},\ \sqrt{-\frac{kc_b}{n}}\right\}$. If $\beta(0) \ne 0$, then Lemma 2.3.5 gives the upper bounds of $\alpha(t)$ and $|\beta(t)|$. Therefore, we complete the proof of Theorem 2.2.7.

Proof of Theorem 2.2.8

From (2.5d),

$$\begin{aligned} \rho' &= -\rho(\sum_{i=1}^{n} \lambda_i) \\ &= -n\rho\lambda_n - \rho\big\{ \sum_{i=1}^{n-1} \lambda_i + (1-n)\lambda_n \big\}. \end{aligned} \tag{2.5.4}$$

The order preserving property of real eigenvalues gives $-\rho\big\{ \sum_{i=1}^{n-1} \lambda_i + (1-n)\lambda_n \big\} \geq 0$. Therefore, ODE system

$$\begin{cases} \rho' = -n\rho\lambda_n - \rho\big\{ \sum_{i=1}^{n-1} \lambda_i + (1-n)\lambda_n \big\}, \\ \lambda_n' = -\lambda_n^2 + \frac{k(\rho - c_b)}{n}, \end{cases} \tag{2.5.5}$$

can be compared with the same system in (2.5.2). Using the global invariant in (2.5.3), we define the blow up region of (2.5.2) by

$$\Omega_2' = \left\{ (x, y) \mid x > 0,\ y < sgn(x - c_b)\sqrt{k\left(\frac{c_b^{\frac{n-2}{n}} x^{\frac{2}{n}}}{n-2} - \frac{2x}{n(n-2)} - \frac{c_b}{n} \right)} \right\}.$$

For any given initial data $(\rho_0, \lambda_{n0}) \in \Omega_2'$ for original system (2.5.5), we can find $\epsilon > 0$ such that initial data $(a(0), b(0)) := (\rho_0 - \epsilon, \lambda_{n0} + \epsilon) \in \Omega_2'$ for system (2.5.2). We know that $a(t) \to \infty$ and $b(t) \to -\infty$ at a finite time. Therefore, by Proposition 2.3.1, the initial data $(\rho_0, \lambda_{n0}) \in \Omega_2'$ will lead to finite time blow up of the original system.

Proof of Theorem 2.2.9

Since $\lambda_{10} = \lambda_{20} = \cdots = \lambda_{n0}$, we have $\lambda_1(t) = \lambda_2(t) = \cdots = \lambda_n(t)$, $\forall\ t \geq 0$. Let $\lambda := \lambda_i$ then (2.2.3) leads to

$$\begin{cases} \lambda' = -\lambda^2 + \frac{k(\rho - c_b)}{n}, \\ \rho' = -n\rho\lambda. \end{cases} \tag{2.5.6}$$

Using the same $q = \lambda^2$ transform as employed in the proof of Theorem 2.2.3 gives the following global invariant

$$\frac{\lambda^2 + \frac{2k}{n(n-2)}\rho + \frac{kc_b}{n}}{\rho^{\frac{2}{n}}} = Const.$$

This ensures the boundedness of both λ and ρ, hence completing the proof of Theorem 2.2.9.

Proof of Theorem 2.2.10

Let $x := \lambda_2 - \lambda_1$ and $y := \lambda_2 + \lambda_1$, then (2.2.3) leads to

$$\begin{cases} x' = -xy, \\ y' = -\frac{y^2}{2} - \frac{x^2}{2} + \frac{k\rho}{2} - \frac{kc_b}{2}. \end{cases} \tag{2.5.7}$$

Suppose $x_0 = \lambda_{20} - \lambda_{10} > 0$. Then $x(t) \neq 0$, $\forall t \geq 0$ and the second equation of (2.5.7) leads to

$$y' = -\frac{y^2}{2} + \left(\frac{k\rho}{x^2} - 1\right)\frac{x^2}{2} - \frac{kc_b}{2}.$$

From $\frac{k\rho_0}{x_0^2} - 1 \leq 0$, we can show that

$$\frac{k\rho}{x^2} - 1 \leq 0, \quad \forall t \geq 0.$$

In fact,

$$\begin{aligned} \left(\frac{k\rho}{x^2} - 1\right)' &= k\left(\frac{\rho'}{x^2} - 2 \cdot \frac{\rho}{x^3} \cdot x'\right) \\ &= \frac{k\rho}{x^2}\{-(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) + 2y\} \\ &= \frac{k\rho}{x^2}\{(\lambda_1 + \lambda_2) - (\lambda_3 + \lambda_4)\} \\ &\leq 0. \end{aligned}$$

Therefore,

$$y' \leq -\frac{y^2}{2} - \frac{kc_b}{2} < -\frac{y^2}{2},$$

which ensures a finite time blow up for any $y_0 \in \mathbb{R}$. This proves Theorem 2.2.10.

CHAPTER 3. FINITE TIME BLOW UP OF SOLUTIONS TO 2D WEAKLY RESTRICTED EULER-POISSON EQUATIONS

Abstract This work proposes weakly restricted Euler-Poisson (WREP) equations as a way to gain a better understanding on Euler-Poisson equations in multiple dimensions. The WREP can be viewed as a slight generalization of the restricted Euler-Poisson (REP) equations, introduced in [H. Liu and E. Tadmor, Comm. Math. Phys. 228 (2002), 435- 466]. We then provide upper-thresholds for finite time blow up of solutions for WREP equations with attractive/repulsive forcing. It is shown that the thresholds depend on the relative size of the initial density and each elements of the initial velocity gradient matrix.

3.1 Introduction and Statement of Main Results

We are concerned with the threshold phenomenon in two dimensional Euler-Poisson equations. The pressure-less Euler-Poisson (EP) equations in multiple dimensions are

$$\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{3.1.1a}$$

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = k \nabla \Delta^{-1}(\rho - c_b), \tag{3.1.1b}$$

which are the usual statements of the conservation of mass and Newton's second law. Here k is a physical constant which parameterizes the repulsive $k > 0$ or attractive $k < 0$ forcing. Also, $c_b > 0$ denotes the constant background state. The local density $\rho = \rho(t, x) : \mathbb{R}^+ \times \mathbb{R}^2 \mapsto \mathbb{R}^+$ and the velocity field $\mathbf{u}(t, x) : \mathbb{R}^+ \times \mathbb{R}^2 \mapsto \mathbb{R}^2$ are the unknowns. This hyperbolic system (3.1.1) with non-local forcing describes the dynamic behavior of many important physical flows, including semi-conductors and plasma physics ($k > 0$) and the collapse of stars due to self gravitation ($k < 0$) [Holm et al. (1981); Makino, T. (1986); Markowich et al. (1990); Brauer et al. (1994); Brenner et al. (1998); Deng et al. (2002)].

To address the fundamental question of the persistence of C^1 regularity for solutions of the Euler-Poisson system and related models, the concept of Critical Threshold was originated and developed in a series of papers by Engelberg, Liu and Tadmor [Engelberg et al. (2001); Liu et al. (2002, 2003, 2004, 2010)] and more. The critical threshold in [Engelberg et al. (2001)] describes the conditional stability of the 1D Euler-Poisson system, where the answer to the question of global vs. local existence depends on whether the initial data crosses a critical threshold. Moving to the multidimensional setup, one has to identify the proper quantities to describe the critical threshold phenomenon. Liu and Tadmor introduce in [Liu et al. (2002)] the method of spectral dynamics which relies on the dynamical system governing eigenvalues of the velocity gradient matrix, $M := \nabla \mathbf{u}$, along particle paths.

We follow their approach and in order to trace the evolution of M, we take the gradient of (3.1.1b) to find

$$\partial_t M + \mathbf{u} \cdot \nabla M + M^2 = k \nabla \otimes \nabla \Delta^{-1}[\rho - c_b] = kR[\rho - c_b], \tag{3.1.2}$$

where R is the Riesz matrix operator, defined as

$$R[f] := \nabla \otimes \nabla \Delta^{-1}[f] = \mathcal{F}^{-1}\left\{ \frac{\xi_j \xi_k}{|\xi|^2} \hat{f}(\xi) \right\}_{j,k=1,2}.$$

Now, the Euler-Poisson equations are recast into the coupled system

$$\begin{aligned} &\frac{d}{dt} M + M^2 = kR[\rho - c_b], \\ &\frac{d}{dt}\rho + \rho \operatorname{tr} M = 0, \end{aligned} \tag{3.1.3}$$

with $\frac{d}{dt}$ standing for the usual convective derivative, $\partial_t + \mathbf{u} \cdot \nabla$. It is the non-local forcing, $kR[\rho - c_b]$, which presents the main obstacle to studying the critical threshold phenomenon of the multidimensional Euler-Poisson equations.

To gain better understanding of the dynamics of velocity gradient M governed by (3.1.3), Liu and Tadmor introduce in [Liu et al. (2002)] the restricted Euler-Poisson (REP) system (3.1.4), which is obtained from (3.1.3) by restricting attention to the local isotropic trace $\frac{k}{n}(\rho - c_b) I_{n \times n}$ of the global coupling term $kR[\rho - c_b]$.

The REP system is given by

$$\begin{aligned} &\frac{d}{dt}M + M^2 = \frac{k}{n}(\rho - c_b)I_{n\times n}, \\ &\frac{d}{dt}\rho + \rho \mathrm{tr} M = 0. \end{aligned} \tag{3.1.4}$$

Replacing the nonlocal forcing term by a local one, it was shown that in the repulsive case, the restricted $2D$ REP model admits a two-sided critical threshold [Liu et al. (2003)]. For arbitrary $n \geq 3$ dimension REP model, the author and Liu identified both upper-thresholds for finite time blow up of solutions and sub-thresholds for global existence of solutions [Lee et al. (2013)].

In this work, we propose 2D weakly restricted Euler-Poisson (WREP) system as a *semi-localized* alternative to (3.1.3). Specifically, we consider a system of (ρ, M), governed by equations for

$$\begin{aligned} &\frac{d}{dt}M + M^2 = \begin{pmatrix} \frac{k}{2}(\rho - c_b) & R_{12} \\ R_{21} & \frac{k}{2}(\rho - c_b) \end{pmatrix}, \\ &\frac{d}{dt}\rho + \rho \mathrm{tr} M = 0, \end{aligned} \tag{3.1.5}$$

subject to initial data $(M, \rho)(0, \cdot) = (M_0, \rho_0)$. We proceed to investigate threshold conditions on the initial data that lead to the finite time blow up of M and ρ. To state our main results, we introduce several quantities with which we characterize the behavior of the velocity gradient tensor M. These are the trace $d := \mathrm{tr}M = \nabla \cdot \mathbf{u}$, the vorticity $\omega := \nabla \times \mathbf{u} = M_{21} - M_{12}$ and $\eta := M_{11} - M_{22}$. Also, we denote $A = -\frac{1}{2}\{\left(\frac{\omega_0}{\rho_0}\right)^2 - \left(\frac{\eta_0}{\rho_0}\right)^2\}$.

Theorem 3.1.1. ***(Repulsive forcing, k >0)*** *Consider the 2-dimensional, repulsive weakly restricted Euler-Poisson system* (3.1.5) *subject to initial data such that* $A > 0$. *Then, the solution of the 2D WREP system will blow up in finite time if one of the following conditions is fulfilled:*

(i) $0 < k < 4Ac_b$ *case: any* $\rho_0 > 0$ *and* $d_0 \in (-\infty, \infty)$.

(ii) $k = 4Ac_b$ *case:* $(\rho_0, d_0) \in S_1 \cup S_2$, *where*

$$S_1 := \{(\rho, d) \mid \rho \leq 2c_b\},$$

$$S_2 := \left\{(\rho, d) \mid \rho > 2c_b \text{ and } d < \sqrt{-\frac{F(2c_b)}{2c_b}\rho + F(\rho)}\right\}.$$

(iii) $k > 4Ac_b$ *case:* $(\rho_0, d_0) \in S_3 \cup S_4$, *where*

$$S_3 := \Big\{(\rho, d) \mid -\frac{F(\rho)}{\rho} + \frac{F(\alpha_1)}{\alpha_1} \geq 0\Big\},$$

$$S_4 := \Big\{(\rho, d) \mid -\frac{F(\rho)}{\rho} + \frac{F(\alpha_1)}{\alpha_1} < 0 \text{ and } d < sgn(\rho - \alpha_1)\sqrt{-\frac{F(\alpha_1)}{\alpha_1}\rho + F(\rho)}\Big\}.$$

Here $\alpha_1 := \frac{k+\sqrt{k^2-4kAc_b}}{2A}$ *and* $F(\rho) := \rho(2A\rho - 2k\log\rho - 2k\frac{c_b}{\rho})$.

Theorem 3.1.2. ***(Attractive forcing, k <0)*** *Consider the 2-dimensional, attractive weakly restricted Euler-Poisson system* (3.1.5) *subject to initial data* ρ_0, $\mathbf{u_0}$. *Then, the solution of the 2D WREP system will blow up in a finite time if the following condition is fulfilled:* $A > 0$ *and*

$$d_0 < sgn(\rho_0 - \alpha_1)\sqrt{-\frac{F(\alpha_1)}{\alpha_1}\rho_0 + F(\rho_0)},$$

where $\alpha_1 := \frac{k+\sqrt{k^2-4kAc_b}}{2A}$ *and* $F(\rho) := \rho(2A\rho - 2k\log\rho - 2k\frac{c_b}{\rho})$.

The following lemma is crucial in our proofs of the main theorems.

Lemma 3.1.3. *From the 2D WREP system*

$$\frac{d}{dt}M + M^2 = \begin{pmatrix} \frac{k}{2}(\rho - c_b) & R_{12} \\ R_{21} & \frac{k}{2}(\rho - c_b) \end{pmatrix}, \quad \frac{d}{dt}\rho + \rho \text{tr} M = 0,$$

it yields the following ordinary differential inequality(ODI) system:

$$\begin{aligned} &d' \leq -\frac{1}{2}d^2 + \frac{1}{2}\Big\{\Big(\frac{\omega_0}{\rho_0}\Big)^2 - \Big(\frac{\eta_0}{\rho_0}\Big)^2\Big\}\rho^2 + k(\rho - c_b), \qquad ' := \frac{d}{dt} \\ &\rho' = -d\rho. \end{aligned} \tag{3.1.6}$$

Proof. From the matrix equation (3.1.5), or

$$\frac{d}{dt}M + \begin{pmatrix} M_{11}^2 + M_{12}M_{21} & dM_{12} \\ dM_{21} & M_{12}M_{21} + M_{22}^2 \end{pmatrix} = \begin{pmatrix} \frac{k}{2}(\rho - c_b) & R_{12} \\ R_{21} & \frac{k}{2}(\rho - c_b) \end{pmatrix},$$

we obtain

$$\begin{cases} \eta' + \eta d = 0, \\ \omega' + \omega d = R_{12} - R_{21} = 0, \\ \rho' + \rho d = 0. \end{cases} \tag{3.1.7}$$

Hence

$$\frac{\eta}{\eta_0} = \frac{\omega}{\omega_0} = \frac{\rho}{\rho_0}.$$

Thus,

$$\begin{aligned}
d' &= -(M_{11}^2 + M_{22}^2) - 2M_{12}M_{21} + k(\rho - c_b) \\
&= -\left\{ M_{11}^2 + M_{22}^2 + \frac{(M_{12}+M_{21})^2}{2} \right\} + \frac{(M_{12}-M_{21})^2}{2} + k(\rho - c_b) \\
&\leq -\frac{(M_{11}+M_{22})^2 + (M_{11}-M_{22})^2}{2} + \frac{1}{2}\omega^2 + k(\rho - c_b) \qquad (3.1.8)\\
&= -\frac{1}{2}d^2 - \frac{1}{2}\eta^2 + \frac{1}{2}\omega^2 + k(\rho - c_b) \\
&= -\frac{1}{2}d^2 + \frac{1}{2}\left\{ \left(\frac{\omega_0}{\rho_0}\right)^2 - \left(\frac{\eta_0}{\rho_0}\right)^2 \right\}\rho^2 + k(\rho - c_b).
\end{aligned}$$

□

Several remarks are in order.

1. Lemma 3.1.3 tells us that one can derive the Riccati-type inequality even when initial vorticity condition, $\nabla \times \mathbf{u}_0(x) \neq 0$. The inequality in (3.1.6) is similar to the Riccati-type inequality in [Chae et al. (2008); Cheng et al. (2009)]. That is, using the inequality

$$d' \leq -\frac{1}{n}d^2 + k(\rho - c_b), \qquad (3.1.9)$$

they proved finite time blow up for C^1 solutions of the full Euler-Poisson equations in $\mathbb{R}^n$ with attractive force, $k < 0$. In their proof, vanishing initial vorticity condition was essential for deriving the Riccati-type inequality in (3.1.9).

2. The critical threshold in the 1D Euler-Poisson system depends only on the relative size of the initial velocity slope and the initial density [Engelberg et al. (2001)]. In contrast to the 1D EP system, the threshold conditions in 2D REP depend on several initial quantities : density ρ_0, divergence $\nabla \cdot \mathbf{u}_0$, vorticity $\nabla \times \mathbf{u}_0$ and gap $v_{0x} - u_{0y}$.

3. The above results show that to ensure the finite time blow up, relatively small absolute value of initial vorticity $|u_{0y} - v_{0x}|$ is needed. This fact is consistent with the results in [Liu et al. (2003)]; the results in [Liu et al. (2003)] show that the global smooth solution is ensured if the initial velocity gradient has complex eigenvalues, which applies, for example, for a class of initial configurations with sufficiently large vorticity.

4. With relatively small initial vorticity, the finite time blow up occurs if the initial divergence is below a threshold, expressed in terms of the initial density and elements in $\nabla \mathbf{u}_0$. So our results can be placed in the perspective of the critical thresholds.

3.2 Proof of The Finite-time Blow-up for 2D WREP with Repulsive Forcing

In this section we show the existence of an upper threshold for the 2D WREP with $k > 0$. Let $A := -\frac{1}{2}\{(\frac{\omega_0}{\rho_0})^2 - (\frac{\eta_0}{\rho_0})^2\}$, then

$$d' \leq -\frac{1}{2}d^2 - A\rho^2 + k(\rho - c_b), \tag{3.2.1a}$$

$$\rho' = -d\rho. \tag{3.2.1b}$$

i) $\mathbf{0 < k \leq 4Ac_b}$ case. (3.2.1a) is rewritten as

$$d' \leq -\frac{1}{2}d^2 - A\left(\rho - \frac{k}{2A}\right)^2 + \frac{k^2}{4A} - kc_b \tag{3.2.2}$$

Since $A > 0$, we have

$$d' \leq -\frac{1}{2}d^2 + \frac{k^2}{4A} - kc_b.$$

So, if $0 < k < 4Ac_b$, i.e. $\frac{k^2}{4A} - k < 0$, the solution exhibits blow up in finite time even if $d_0 > 0$. This completes the proof of part (i) of Theorem 3.1.1.

ii) $\mathbf{k > 4Ac_b}$ case. We first consider the corresponding ODE system

$$\begin{aligned} e' &= -\frac{1}{2}e^2 - A\zeta^2 + k(\zeta - c_b), \\ \zeta' &= -e\zeta. \end{aligned} \tag{3.2.3}$$

Note that the ODE system admits two distinct critical points, i.e. $(\zeta, e) = (\alpha_i, 0)$ where,

$$(\alpha_1, 0) := \left(\frac{k + \sqrt{k^2 - 4kAc_b}}{2A}, 0\right), \quad (\alpha_2, 0) := \left(\frac{k - \sqrt{k^2 - 4kAc_b}}{2A}, 0\right),$$

and that $(\rho, d) = (\alpha_1, 0)$ is a saddle and $(\rho, d) = (\alpha_2, 0)$ is a spiral. We shall use the above facts to construct the critical threshold via the phase plane analysis. Following the same q-transformation as that employed in [Liu et al. (2003)], we set $q := e^2$ and differentiate along the particle path $\{(t, X(t,a)) \mid X_t(t,a) = u(t, X(t,a)),\ X(t=0,a) = a\}$, and we get

$$\frac{dq}{d\zeta} = 2e\frac{e'}{\zeta'} = \frac{q}{\zeta} + 2A\zeta - 2k(1 - \frac{c_b}{\zeta}),$$

which yields

$$\frac{d}{d\zeta}\left(\frac{q}{\zeta}\right) = 2A - 2k\left(\frac{1}{\zeta} - \frac{c_b}{\zeta^2}\right). \tag{3.2.4}$$

Integration leads to a global invariant

$$\frac{e^2}{\zeta} - \frac{e_*^2}{\zeta_*} = -2\int_{\zeta_*}^{\zeta} \frac{-As^2 + k(s - c_b)}{s^2}\, ds, \tag{3.2.5}$$

where e_* and ζ_* are some constants.

By setting $(\zeta_*, e_*) = (\alpha_1, 0)$, we find the separatrix curve passing through $(\alpha_1, 0)$,

$$\frac{e^2}{\zeta} = -2\int_{\alpha_1}^{\zeta} \frac{-As^2 + k(s - c_b)}{s^2}\, ds. \tag{3.2.6}$$

The above curve has two x intercepts. One is $(\alpha_1, 0)$ and the other is denoted by $(\alpha_3, 0)$ with $0 < \alpha_3 < \alpha_2$.

In fact, consider

$$\int_{\zeta}^{\alpha_1} \frac{-As^2 + k(s - c_b)}{s^2}\, ds = \int_{\zeta}^{\alpha_2} \frac{-As^2 + k(s - c_b)}{s^2}\, ds + \int_{\alpha_2}^{\alpha_1} \frac{-As^2 + k(s - c_b)}{s^2}\, ds.$$

Note that $-As^2 + k(s - c_b) \geq 0$, $\forall s \in [\alpha_2, \alpha_1]$ and $\lim_{\zeta\to 0+} \int_{\zeta}^{\alpha_2} \frac{-As^2+k(s-c_b)}{s^2}\, ds \to -\infty$. This proves the existence of intercept $(\alpha_3, 0)$ and the following identity,

$$\int_{\alpha_3}^{\alpha_1} \frac{-As^2 + k(s - c_b)}{s^2}\, ds = 0. \tag{3.2.7}$$

Back to ODI system (3.2.1), the same q-transformation gives us

$$\frac{d}{d\rho}\left(\frac{q}{\rho}\right) \geq 2A - 2k\left(\frac{1}{\rho} - \frac{c_b}{\rho^2}\right). \tag{3.2.8}$$

We now discuss subcases distinguished by the location of initial points; see Figure 3.1.

- $(\rho_0, d_0) \in \Omega_1$, where

$$\Omega_1 := \left\{(\rho, d)\middle|\ \alpha_3 \leq \rho \leq \alpha_1,\ d < -\sqrt{2\rho\int_{\rho}^{\alpha_1} \frac{-As^2 + k(s - c_b)}{s^2}\, ds}\right\}.$$

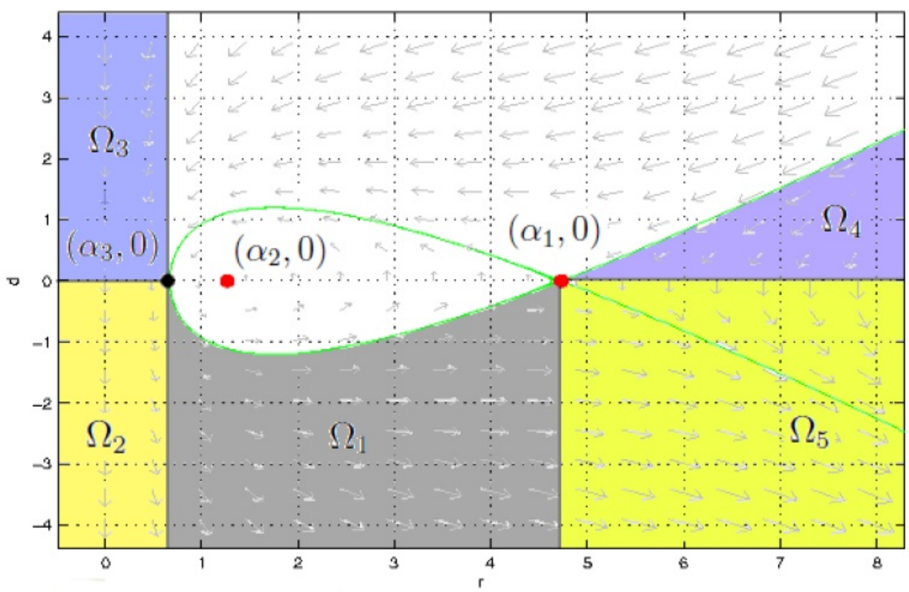

Figure 3.1 The blow up region of $k > 4Ac_b$ case.

First, we show no orbit of the ODI touches the lower left arc of the separatrix curve: (3.2.8) leads to

$$\frac{d^2}{\rho} - \frac{d_0^2}{\rho_0} \geq -2\int_{\rho_0}^{\rho} \frac{-As^2 + k(s - c_b)}{s^2}\, ds. \tag{3.2.9}$$

Now, consider a point (ρ_0, d_*) on the lower left arc of the separatrix curve, i.e.,

$$\frac{d_*^2}{\rho_0} = -2\int_{\alpha_1}^{\rho_0} \frac{-As^2 + k(s - c_b)}{s^2}\, ds.$$

Since $d_0 < d_* < 0$ we have

$$\begin{aligned} \frac{d^2}{\rho} &\geq \frac{d_0^2}{\rho_0} - 2\int_{\rho_0}^{\rho} \frac{-As^2 + k(s - c_b)}{s^2}\, ds \\ &> \frac{d_*^2}{\rho_0} - 2\int_{\rho_0}^{\rho} \frac{-As^2 + k(s - c_b)}{s^2}\, ds \\ &= -2\int_{\alpha_1}^{\rho} \frac{-As^2 + k(s - c_b)}{s^2}\, ds \end{aligned}$$

Hence, as long as $(\rho_0, d_0) \in \Omega_1$, no orbit of the ODI touches the lower left arc of the separatirx curve.

Next, we show that if $(\rho_0, d_0) \in \Omega_1$ then $(\rho(t), d(t)) \nrightarrow (\alpha_1, 0)$ as $t \to \infty$. Suppose $\rho(t) \nearrow \alpha_1$ and $d(t) \nearrow 0$ as $t \to \infty$. Then as $t \to \infty$, (3.2.9) leads to

$$-\frac{d_0^2}{\rho_0} \geq -2\int_{\rho_0}^{\alpha_1} \frac{-As^2 + k(s - c_b)}{s^2}\, ds.$$

But this contradicts the fact that $(\rho_0, d_0) \in \Omega_1$. Finally, we show that if $(\rho_0, d_0) \in \Omega_1$ then $\exists t^* < \infty$ such that $\rho(t^*) > \alpha_1$. Suppose $\alpha_3 \leq \rho(t) \leq \alpha_1$, $\forall t > 0$. Then, since $d(t) < 0$, from $\rho(t) = \rho_0 \exp(-\int_0^t d(s)ds)$, $d(t)$ must go to 0. Since no orbit can touch the lower left arc, we are left with only one possibility $(\rho(t), d(t)) \to (\alpha_1, 0)$. But this contradicts the second argument. Hence $\rho(t) > \alpha_1$ in finite time t^*.

- $(\rho_0, d_0) \in \Omega_2 := \{(\rho, d) | 0 < \rho < \alpha_3,\ d < 0\}$. We will show that if $(\rho_0, d_0) \in \Omega_2$, then $(\rho(t), d(t)) \in \Omega_1$ in finite time. Suppose not, i.e. suppose $\rho(t) < \alpha_3$, $\forall t > 0$. Then

$$d' < -\frac{1}{2}d^2 - K, \quad \forall t > 0 \text{ where } K := A(\alpha_3 - \alpha_1)(\alpha_3 - \alpha_2) > 0.$$

That is, $d' < -K$, $\forall t > 0$, which upon integration over $[0, t]$ gives

$$d \leq d_0 - Kt.$$

This tells us that

$$-\int_0^t d(s)ds \geq -d_0 t + \frac{Kt^2}{2}$$

and hence

$$\rho(t) = \rho_0 \exp\left(-\int_0^t d(s)ds\right) \geq \rho_0 \exp\left(-d_0 t + \frac{Kt^2}{2}\right).$$

But, since $d_0 < 0$ and $K > 0$, $\rho(t) \geq \alpha_3$ in finite time. Therefore, we get the contradiction.

- $(\rho_0, d_0) \in \Omega_3 := \{(\rho, d) | 0 < \rho \leq \alpha_3 \text{ and } d \geq 0\}$. As long as $(\rho(t), d(t)) \in \Omega_3$, ρ is decreasing and

$$d' < -\frac{1}{2}d^2 - K \leq -K,$$

where $K := A(\alpha_3 - \alpha_1)(\alpha_3 - \alpha_2)$. Therefore, $(\rho_0, d_0) \in \Omega_3$ implies $(\rho(t), d(t)) \in \Omega_2$ in finite time t.

- $(\rho_0, d_0) \in \Omega_4$, where

$$\Omega_4 := \left\{(\rho, d)\middle|\ \rho > \alpha_1,\ 0 \le d < \sqrt{-2\rho\int_{\alpha_1}^{\rho} \frac{-As^2 + k(s - c_b)}{s^2}\, ds}\right\}.$$

First, we show no orbit of the ODI touches the upper right branch of the separatrix curve: Note that in Ω_4, since $\rho, d > 0$, we have $\rho' \le 0$. Thus (3.2.8) leads to

$$\frac{d^2}{\rho} - \frac{d_0^2}{\rho_0} \le -2\int_{\rho_0}^{\rho} \frac{-As^2 + k(s - c_b)}{s^2}\, ds. \tag{3.2.10}$$

Now, consider a point (ρ_0, d^*) on the invariant, i.e.,

$$\frac{d^{*2}}{\rho_0} = -2\int_{\alpha_1}^{\rho_0} \frac{-As^2 + k(s - c_b)}{s^2}\, ds.$$

Since $0 < d_0 < d_*$ we have

$$\begin{aligned}
\frac{d^2}{\rho} &\le \frac{d_0^2}{\rho_0} - 2\int_{\rho_0}^{\rho} \frac{-As^2 + k(s - c_b)}{s^2}\, ds \\
&< \frac{d_*^2}{\rho_0} - 2\int_{\rho_0}^{\rho} \frac{-As^2 + k(s - c_b)}{s^2}\, ds \\
&= -2\int_{\alpha_1}^{\rho} \frac{-As^2 + k(s - c_b)}{s^2}\, ds
\end{aligned}$$

Hence, as long as $(\rho_0, d_0) \in \Omega_4$, no orbit of the ODI touches the upper right branch of the separatirx curve.

Next, we show that if $(\rho_0, d_0) \in \Omega_4$ then $(\rho(t), d(t)) \nrightarrow (\alpha_1, 0)$ as $t \to \infty$. Suppose $\rho(t) \searrow \alpha_1$ and $d(t) \searrow 0$ as $t \to \infty$. Then as $t \to \infty$, (3.2.10) leads to

$$-\frac{d_0^2}{\rho_0} \le -2\int_{\rho_0}^{\alpha_1} \frac{-As^2 + k(s - c_b)}{s^2}\, ds.$$

But this contradicts the fact that $(\rho_0, d_0) \in \Omega_4$.

Finally, due to non-touching result and the fact that $\lim_{t\to\infty}(\rho, d) \neq (\alpha_1, 0)$, any orbit starting from within Ω_4 must enter $\Omega_5 := \{(\rho, d)|\rho > \alpha_1 \text{ and } d < 0\}$ through $d = 0$ and $\rho > \alpha_1$. To sum up, we arrive at the following observation.

Lemma 3.2.1. *If $(\rho_0, d_0) \in \bigcup_{i=1}^4 \Omega_i$, then $(\rho(t), d(t)) \in \Omega_5$ in finite time, where*

$$\Omega_5 := \{(\rho, d)|\ \alpha_1 < \rho \text{ and } d < 0\}.$$

Now in Ω_5, we shall carry out the blow up analysis of

$$\begin{cases} d' \le -\frac{1}{2}d^2 - A\rho^2 + k(\rho - c_b) = -\frac{1}{2}d^2 - A(\rho-\alpha_1)(\rho-\alpha_2), \\ \rho' = -d\rho, \end{cases} \tag{3.2.11}$$

through a comparison with the corresponding ODE system

$$\begin{cases} e' = -\frac{1}{2}e^2 - A\zeta^2 + k(\zeta - c_b) = -\frac{1}{2}e^2 - A(\zeta-\alpha_1)(\zeta-\alpha_2), \\ \zeta' = -e\zeta. \end{cases} \tag{3.2.12}$$

The following lemma shows the monotonicity relation between the ODE and the ODI in Ω_5.

Lemma 3.2.2. $\begin{cases} d(0) < e(0) < 0 \\ \zeta(0) < \rho(0), \end{cases}$ *implies* $\begin{cases} d(t) < e(t) < 0 \\ \zeta(t) < \rho(t), \end{cases}$ $\forall t \ge 0,$ *as long as* $\zeta(t) > \alpha_1$, $\forall t \ge 0$.

Proof. It can be proved by contradiction. Suppose t_1 is the earliest time when the above assertion is violated. Then

$$\zeta(t_1) = \zeta(0)e^{-\int_0^{t_1} e(t)dt} < \rho(0)e^{-\int_0^{t_1} d(t)dt} = \rho(t_1).$$

Therefore, we are left with only one possibility $e(t_1) = d(t_1)$.

From (3.2.11) and (3.2.12),

$$(e-d)' \ge -\frac{1}{2}(e^2 - d^2) - A\big\{(\zeta-\alpha_1)(\zeta-\alpha_2) - (\rho-\alpha_1)(\rho-\alpha_2)\big\}. \tag{3.2.13}$$

Since $e(t) - d(t) > 0$ for $t < t_1$ and $e(t_1) - d(t_1) = 0$, hence at $t = t_1$ we have

$$(e(t_1) - d(t_1))' \le 0.$$

But, since $\rho(t_1) > \zeta(t_1)$, when (3.2.13) is evaluated at $t = t_1$ gives

$$-A\big\{(\zeta(t_1)-\alpha_1)(\zeta(t_1)-\alpha_2) - (\rho(t_1)-\alpha_1)(\rho(t_1)-\alpha_2)\big\} > 0.$$

This leads to a contradiction, as needed. $\square$

The following lemma provides the blow up conditions of the modified system in (3.2.12), which in turn, will lead to the blow up of the original system in (3.2.11).

Lemma 3.2.3. *Consider the modified system* (3.2.12)*, equipped with initial data* (ζ_0, e_0)*. If* $(\zeta_0, e_0) \in \Omega_5$*, then* $e \to -\infty$*,* $\zeta \to \infty$ *at a finite time.*

Proof. Consider

$$\begin{cases} e' = -\frac{1}{2}e^2 - A(\zeta - \alpha_1)(\zeta - \alpha_2), \\ \zeta' = -e\zeta. \end{cases}$$

Note that if $(\zeta_0, e_0) \in \Omega_5$, then $\zeta(t)$ is increasing $\forall t$. Thus, $\zeta(t) > \alpha_1$, $\forall t$. This implies $e' < -\frac{1}{2}e^2$, which upon integration yields

$$e(t) < \frac{2e_0}{te_0 + 2}.$$

This implies that

$$e(t) \to -\infty \quad \text{and} \quad \zeta(t) = \zeta_0 \exp\left(- \int_0^t e(s)ds \right) \to \infty \quad \text{as } t \to t^*$$

with the blow up time $t^* < -\frac{2}{e_0}$. □

Now we are ready for the last step of proving part (iii) of Theorem 3.1.1. We combine the monotonicity relation in Lemma 3.2.2 with Lemma 3.2.1 and Lemma 3.2.3. Consider any given initial data $(\rho_0, d_0) \in \Omega_5$ for the ODI (3.2.1). Since Ω_5 is an open set, so we can find $\epsilon > 0$ such that $(\rho_0 - \epsilon, d_0 + \epsilon) \in \Omega_5$. We set this latter data as an initial data of the ODE (3.2.12) for the comparison purpose. This latter initial data will lead to finite time blow up of the ODE and thus initial data $(\rho_0, d_0) \in \Omega_5$ will lead to finite time blow up of the ODI. Furthermore, by Lemma 3.2.2, we know that if an initial data of the ODI is contained in $\Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$, then $(\rho(t), d(t)) \in \Omega_5$ in finite time. Hence, initial data $(\rho_0, d_0) \in \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4 \cup \Omega_5$ will lead to finite time blow up of the original ODI.

To sum up, the above arguments give us the upper thresholds which lead to finite-time breakdown of solutions to the WREP equation. The threshold curve can be expressed as an union of two sets: One is half straight line $\{(\rho, d) \mid \rho = \alpha_3,\ d > 0\}$, and the other is a union of the lower-left arc and upper-right branches of the separatrix curve $\frac{d^2}{\rho} = -2\int_{\alpha_1}^{\rho} \frac{-As^2 + k(s - c_b)}{s^2}\, ds$, i.e.,

$$\left\{ (\rho, d) \;\middle|\; \rho \geq \alpha_3,\ d = sgn(\rho - \alpha_1)\sqrt{-2\rho \int_{\alpha_1}^{\rho} \frac{-As^2 + k(s - c_b)}{s^2}\, ds} \right\}.$$

Expanding the above integral and using the identity in (3.2.7) completes the proof of part (iii) of Theorem 3.1.1.

iii) $\mathbf{k = 4Ac_b}$ case. The system in (3.2.1) is rewritten as

$$d' \leq -\frac{1}{2}d^2 - A(\rho - 2c_b)^2, \tag{3.2.14a}$$

$$\rho' = -d\rho. \tag{3.2.14b}$$

Again, we shall carry out the blow up analysis of the above system through a comparison with the corresponding ODE system

$$e' \leq -\frac{1}{2}e^2 - A(\zeta - 2c_b)^2, \tag{3.2.15a}$$

$$\zeta' = -e\zeta. \tag{3.2.15b}$$

Note that the above ODE system admits one critical point, i.e.,

$$(\zeta, e) = (2c_b, 0)$$

and that $(2c_b, 0)$ is a saddle. As we did in the previous case, we find the separatrix curve passing $(2c_b, 0)$,

$$\frac{e^2}{\zeta} = -2\int_{2c_b}^{\zeta} \frac{-A(s - 2c_b)^2}{s^2}\, ds.$$

We now discuss subcases distinguished by the location of initial points; see Figure 3.2.

- $(\rho_0, d_0) \in \Omega_3 := \{(\rho, d)|\ 0 < \rho < 2c_b \text{ and } d \geq 0\}$. We show that $(\rho_0, d_0) \in \Omega_3$ implies $(\rho(t), d(t)) \in \Omega_1$ in finite time. Here $\Omega_1 := \{(\rho, d)|\ \rho > 0, \quad d < 0\}$. Indeed, as long as $(\rho(t), d(t)) \in \Omega_3$, ρ is decreasing and $d' \leq -\frac{1}{2}d^2 - A(\rho - 2c_b)^2 < -A(\rho_0 - 2c_b)^2$. Therefore, $(\rho_0, d_0) \in \Omega_3$ implies $(\rho(t), d(t)) \in \Omega_1$ in finite time.

- $(\rho_0, d_0) \in \Omega_2 := \left\{(\rho, d) \middle|\ \rho > 2c_b,\ 0 \leq d < \sqrt{-2\rho \int_{2c_b}^{\rho} \frac{-A(\rho - 2c_b)^2}{s^2}\, ds}\right\}$. Similar to the case of $k > 4Ac_b$, due to non-touching argument and the fact that $\lim_{t\to\infty}(\rho, d) \neq (2c_b, 0)$, any orbit starting from within Ω_2 must enter Ω_1 through $d = 0$ and $\rho > 2c_b$ in finite time.

Finally, since $(\rho_0, d_0) \in \Omega_2 \cup \Omega_3$ implies $(\rho(t), d(t)) \in \Omega_1$ in finite time, it suffices to show finite time blow up of $\rho(t)$ and $d(t)$ with initial data $(\rho_0, d_0) \in \Omega_1$. We state this as a lemma.

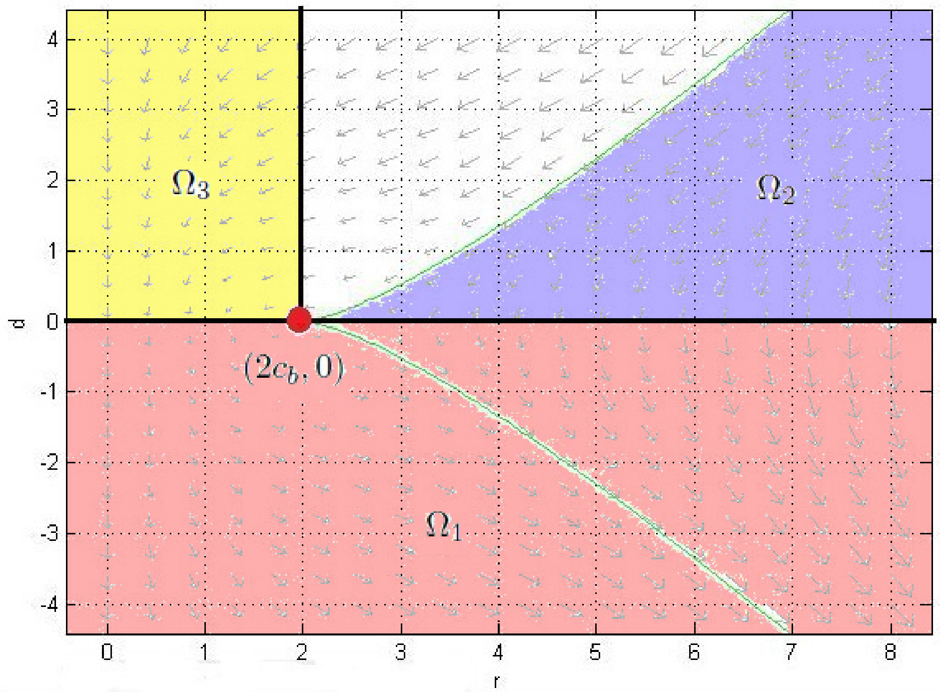

Figure 3.2 The blow up region of $k = 4Ac_b$ case.

Lemma 3.2.4. *Consider the system* (3.2.14)*, equipped with initial data* (ρ_0, d_0)*. If* $(\rho_0, d_0) \in \Omega_1$*, then* $d \to -\infty$*,* $\rho \to \infty$ *at a finite time.*

Proof. Consider

$$\begin{cases} d' = -\frac{1}{2}d^2 - A(\rho - 2c_b)^2, \\ \rho' = -d\rho. \end{cases}$$

Note that since $A > 0$, we obtain

$$d' \leq -\frac{1}{2}d^2,$$

which upon integration yields

$$d(t) \leq \frac{2d_0}{td_0 + 2}.$$

This implies that

$$d(t) \to -\infty \quad \text{and} \quad \rho(t) = \rho_0 \exp\left(-\int_0^t d(s)ds \right) \to \infty \quad \text{as } t \to t^*$$

with the blow up time $t^* < -\frac{2}{d_0}$. □

The above arguments give us the upper thresholds with lead to finite-time breakdown of the WREP equation. The threshold curve can be expressed as a union of two sets: One is half straight line

$$\{(\rho, d) \mid \rho = 2c_b,\ d > 0\},$$

and the other is the upper branches of the separatrix curve $\frac{d^2}{\rho} = -2\int_{2c_b}^{\rho} \frac{-A(s-2c_b)^2}{s^2}\,ds$. i.e.,

$$\left\{(\rho, d) \;\middle|\; \rho \geq 2c_b,\ d = \sqrt{-2\rho\int_{2c_b}^{\rho} \frac{-A(s-2c_b)^2}{s^2}\,ds}\right\}.$$

Expanding the above integral completes the proof of part (ii) of Theorem 3.1.1.

3.3 Proof of The Finite-time Blow-up for 2D WREP with Attractive Forcing

In this section we prove the existence of a one-sided threshold condition which leads to finite-time breakdown of the 2D WREP with attractive forcing($k < 0$). We shall carry out the blow up analysis of

$$\begin{aligned} d' &\leq -\frac{1}{2}d^2 + \frac{1}{2}\left\{\left(\frac{\omega_0}{\rho_0}\right)^2 - \left(\frac{\eta_0}{\rho_0}\right)^2\right\}\rho^2 + k(\rho - c_b), \\ \rho' &= -d\rho, \end{aligned} \tag{3.3.1}$$

through a comparison with the corresponding ODE system

$$\begin{aligned} e' &= -\frac{1}{2}e^2 + \frac{1}{2}\left\{\left(\frac{\omega_0}{\rho_0}\right)^2 - \left(\frac{\eta_0}{\rho_0}\right)^2\right\}\zeta^2 + k(\zeta - c_b), \\ \zeta' &= -e\zeta. \end{aligned} \tag{3.3.2}$$

As we did before, let $A := -\frac{1}{2}\left\{\left(\frac{\omega_0}{\rho_0}\right)^2 - \left(\frac{\eta_0}{\rho_0}\right)^2\right\} > 0$ and for simplicity we set $c_b = 1$. The following lemma shows the monotonicity relation between (3.3.1) and (3.3.2). The proof is similar to that in [Cheng et al. (2009)], so details are omitted.

Lemma 3.3.1. $\begin{cases} d(0) < e(0) \\ 0 < \zeta(0) < \rho(0) \end{cases}$ *implies* $\begin{cases} d(t) < e(t) \\ 0 < \zeta(t) < \rho(t) \end{cases}$ *for* $t \geq 0$, *as long as all solutions remain bounded.*

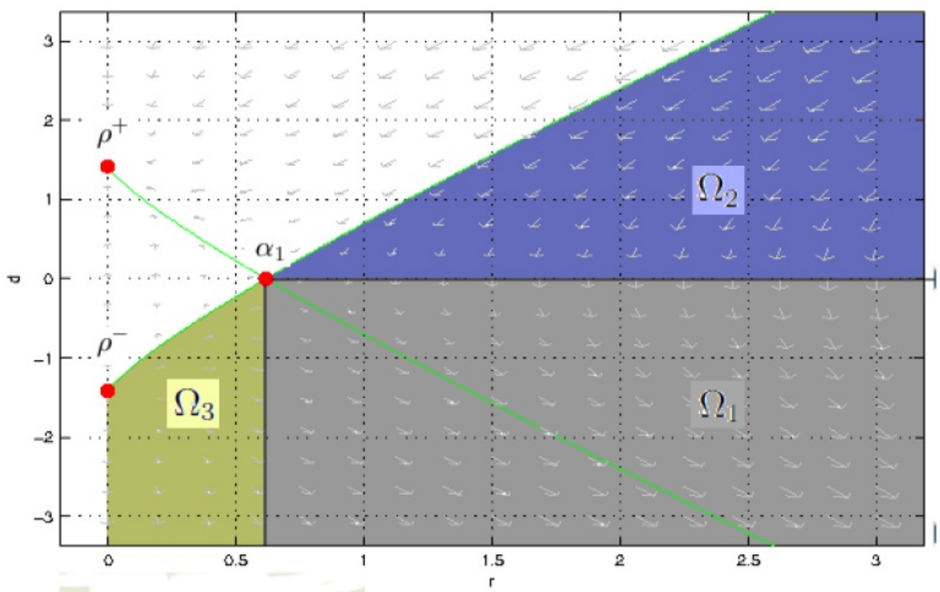

Figure 3.3 The blow up region of $k < 0$ case.

System (3.3.2) admits three distinct critical points:

$$(\rho^{\pm}, d^{\pm}) := (0, \pm\sqrt{-2k}), \quad (\alpha_1, 0) := \left(\frac{k + \sqrt{k^2 - 4Ak}}{2A}, 0\right)$$

and that (ρ^+, d^+) is a nodal sink, (ρ^-, d^-) is a nodal source and $(\alpha_1, 0)$ is a saddle point. Also, as we did in the previous section, the separatrix curve passing $(\alpha_1, 0)$ is given by

$$\frac{d^2}{\rho} = -2\int_{\alpha_1}^{\rho} \frac{-As^2 + k(s-1)}{s^2}\, ds.$$

Note that the comparison principle in Lemma 3.3.1 applies in $\Omega_1 := \{(\rho, d) | \rho > \alpha_1 \text{ and } d < 0\}$. We now discuss subcases distinguished by the location of initial points;

- $(\rho_0, d_0) \in \Omega_2$, where

$$\Omega_2 := \left\{(\rho, d)|\ \alpha_1 < \rho,\ 0 \le d < \sqrt{-2\rho\int_{\alpha_1}^{\rho} \frac{-As^2 + k(s-1)}{s^2}\, ds}\right\}.$$

  Due to non-touching result we showed in the previous section and the fact that $\lim_{t\to\infty}(\rho, d) \nrightarrow (\alpha_1, 0)$, we know that if $(\rho_0, d_0) \in \Omega_2$, then $(\rho(t), d(t)) \in \Omega_1$ in finite time. The proof of this is the same as that in Ω_4 of the repulsive case.

- $(\rho_0, d_0) \in \Omega_3$, where

$$\Omega_3 := \left\{(\rho, d)|\ 0 < \rho \leq \alpha_1,\ d < -\sqrt{-2\rho \int_{\alpha_1}^{\rho} \frac{-As^2 + k(s-1)}{s^2}\, ds}\right\}.$$

As we did in repulsive case, the non-touching result and the fact that $\lim_{t\to\infty}(\rho, d) \nrightarrow (\alpha_1, 0)$ can be applied here too. We know that $\lim_{t\to\infty} d(t) \nrightarrow 0$. Thus

$$\rho(t) = \rho_0 \exp(-\int_0^t d(s)ds) > \alpha_1,$$

in finite time.

To sum up, we arrive at the following observation.

Lemma 3.3.2. *If $(\rho_0, d_0) \in \Omega_2 \cup \Omega_3$, then $(\rho(t), d(t)) \in \Omega_1$ in finite time.*

The following lemma provides the blow up conditions of the modified system in (3.3.2), which in turn, will lead to the blow up of the original system in (3.3.1).

Lemma 3.3.3. *Consider the modified system* (3.3.2)*, equipped with initial data (ζ_0, e_0). If $(\zeta_0, e_0) \in \Omega_1$, then $e \to -\infty$, $\zeta \to \infty$ at a finite time.*

Proof. Consider

$$\begin{cases} e' = -\frac{1}{2}e^2 - A(\zeta - \alpha_1)(\zeta - \beta), \\ \zeta' = -e\zeta. \end{cases}$$

where $\beta = \frac{k-\sqrt{k^2-4Ak}}{2A} < 0$. Note that if $(\zeta_0, e_0) \in \Omega_1$, then $\zeta(t)$ is increasing in t. Thus, $\zeta(t) > \alpha_1,\ \forall t$. This implies $e' < -\frac{1}{2}e^2$, which upon integration yields

$$e(t) < \frac{2e_0}{te_0 + 2}.$$

Hence, the blow up time t^* of $e(t)$ must satisfy

$$t^* < -\frac{2}{e_0}.$$

Also, $e \to -\infty$ and $\zeta = \zeta_0 \exp(-\int_0^t e(s)ds) \to \infty$. $\square$

Now we are ready for the last step of proving Theorem 3.1.2. We combine the monotonicity relation in Lemma 3.3.1 with Lemma 3.3.2 and Lemma 3.3.3. Consider given any initial data

$(\rho_0, d_0) \in \Omega_1$ for the ODI in (3.3.1). Since Ω_1 is an open set, we can find $\epsilon > 0$ such that $(\rho_0 - \epsilon, d_0 + \epsilon) \in \Omega_1$. We set this latter data as an initial data of the ODE in (3.3.2) for the comparison purpose. This latter initial data will lead to finite time blow up of the ODE and thus initial data $(\rho_0, d_0) \in \Omega_1$ will lead to finite time blow up of the ODI. Furthermore, by Lemma 3.3.2, we know that if an initial data of the ODI is contained in $\Omega_2 \cup \Omega_3$, then $(\rho(t), d(t)) \in \Omega_1$ in finite time. Hence, initial data

$$(\rho_0, d_0) \in \Omega_1 \cup \Omega_2 \cup \Omega_3$$

will lead to finite time blow up of the original ODI.

We close this section by stating the upper thresholds which determine the blow up region of the WREP equation. The upper right and lower left branches of

$$\frac{d^2}{\rho} = -2 \int_{\alpha_1}^{\rho} \frac{-As^2 + k(s-1)}{s^2}\, ds$$

are the critical thresholds. So the upper thresholds can be expressed as

$$\left\{ (\rho, d)\ |\rho > 0, \quad d = sgn(\rho - \alpha_1)\sqrt{-2\rho \int_{\alpha_1}^{\rho} \frac{-As^2 + k(s-1)}{s^2}\, ds} \right\}.$$

This completes the proof of Theorem 3.1.2.

CHAPTER 4. THRESHOLDS FOR SHOCK FORMATION IN TRAFFIC FLOW MODELS WITH ARRHENIUS LOOK-AHEAD DYNAMICS

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Abstract We investigate a class of nonlocal conservation laws with the nonlinear advection coupling both local and nonlocal mechanism, which arises in several applications such as the collective motion of cells and traffic flows. It is proved that the C^1 solution regularity of this class of conservation laws will persist at least for a short time. This persistency may continue as long as the solution gradient remains bounded. Based on this result, we further identify sub-thresholds for finite time shock formation in traffic flow models with Arrhenius look-ahead dynamics.

4.1 Introduction

In this work we investigate a class of nonlocal conservation laws,

$$\begin{cases} \partial_t u + \partial_x F(u, \bar{u}) = 0, & t > 0, x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{4.1.1}$$

where u is the unknown, F is a given smooth function, and $\bar{u}$ is given by

$$\bar{u}(t, x) = (K * u)(t, x) = \int_{\mathbb{R}} K(x - y) u(t, y)\, dy, \tag{4.1.2}$$

where K is assumed in $W^{1,1}(\mathbb{R})$. The advection couples both local and nonlocal mechanism. This class of conservation laws appears in several applications including traffic flows [Kurganov et al. (2009); Sopasakis et al. (2006)], the collective motion of biological cells [Dolak et al.

(2005); Burger et al. (2008); Perthame et al. (2009)], dispersive water waves [Whitham (1974); Holm et al. (2005); Degasperis et al. (1999); Liu (2006)], the radiating gas motion [Hamer (1971); Rosenau (1989); Liu et al. (2001)], high-frequency waves in relaxing medium [Hunter (1990); Parkes (2002); Vakhnenko (1992)], and the kinematic sedimentation [Kynch (1952); Zumbrun (1999); Karlsen et al. (2011)].

We are interested in the persistence of the C^1 solution regularity for (4.1.1). As is known that the typical well-posedness result asserts that either a solution of a time-dependent PDE exists for all time or else there is a finite time such that some norm of the solution becomes unbounded as the life span is approached. The natural question is whether there is a critical threshold for the initial data such that the persistence of the C^1 solution regularity depends only on crossing such a critical threshold. This concept of critical threshold and associated methodology is originated and developed in a series of papers by Engelberg, Liu and Tadmor [Engelberg et al. (2001); Liu et al. (2002, 2003)] for a class of Euler-Poisson equations.

In this paper we attempt to study such a critical phenomena in (4.1.1). C^1 solution regularity is shown to persist at least for finite time. Moreover, such persistency may continue as long as the solution gradient remains bounded. We also identify sub-thresholds for finite time shock formation in some special traffic flow models, as well as (4.1.1) with one sided interaction kernels. These together partially confirm the critical threshold phenomenon in non-local conservation laws (4.1.1).

The traffic flow model that motivated this study is the one with looking ahead relaxation introduced by Sopasakis and Katsoulakis [Sopasakis et al. (2006)]:

$$\begin{cases} \partial_t u + \partial_x (u(1-u)e^{-K*u}) = 0, & t > 0, x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{4.1.3}$$

where $u(t,x)$ represents a vehicle density normalized in the interval $[0,1]$ and the relaxation kernel

$$K(r) = \begin{cases} \frac{K_0}{\gamma}, & \text{if } -\gamma \le r \le 0, \\ 0, & \text{otherwise}, \end{cases} \tag{4.1.4}$$

is the constant interaction potential, where γ is a positive constant proportional to the look-ahead distance and K_0 is a positive interaction strength. We set $K_0 = 1$.

An improved interaction potential for (4.1.3) is introduced in [Kurganov et al. (2009)] with

$$K(r) = \begin{cases} \frac{2}{\gamma}\left(1+\frac{r}{\gamma}\right), & -\gamma \le r \le 0, \\ 0, & \text{otherwise}. \end{cases} \tag{4.1.5}$$

This linear potential is intended to take into account the fact that a car's speed is affected more by nearby vehicles than distant ones. The authors in [Kurganov et al. (2009)] carried out some careful numerical study of the traffic flow model (4.1.3), through three examples: red light traffic, traffic jam on a busy freeway and a numerical breakdown study. In the case of a good visibility (large γ), their numerical studies suggest that (4.1.3) with the modified potential (4.1.5) yields solutions that seem to better correspond to reality.

The objective of this article is therefore twofold : i) to establish local wellposedness of smooth solutions for (4.1.1); ii) to identify threshold conditions for the finite time shock formation of the traffic flow model (4.1.3) subject to two different potentials (4.1.4) and (4.1.5), respectively. The finite time shock formation of solutions in traffic flows are understood as congestion formation.

We use X to denote a space $X(\mathbb{R})$ for $X = H^2(= W^{2,2})$ or $W^{1,1}$, where $W^{k,p}$ denotes a standard Sobolev space. The main results are collectively stated as follows.

Theorem 4.1.1. *(**Local existence**) Suppose $F \in C^3(\mathbb{R},\mathbb{R})$ and $K \in W^{1,1}$. If $u_0 \in H^2$, or $u_0 \in L^\infty$ and $u_{0x} \in H^1$, then there exists $T > 0$, depending on the data, such that* (4.1.1) *admits a unique solution $u \in C^1([0,T) \times \mathbb{R})$. Moreover, if the maximum life span $T^* < \infty$, then*

$$\lim_{t \to T^*-} \|\partial_x u(t,\cdot)\|_{L^\infty} = \infty.$$

Theorem 4.1.2. *Consider* (4.1.3) *with constant potential* (4.1.4)*. Suppose that $u_0 \in H^2$ and $0 \le u_0(x) \le 1$ for all $x \in \mathbb{R}$. If*

$$\sup_{x\in\mathbb{R}}[u_0'(x)] > \frac{1}{\gamma}\left(\frac{1}{2} + \frac{\sqrt{2}}{4}\cdot\sqrt{3 - \min\left\{-1,\ \gamma\cdot\inf_{x\in\mathbb{R}}[u_0'(x)]\right\}}\right), \tag{4.1.6}$$

then u_x must blow up at some finite time.

Theorem 4.1.3. *Consider* (4.1.3) *with linear potential* (4.1.5)*. Suppose that* $u_0 \in H^2$ *and* $0 \leq u_0(x) \leq 1$ *for all* $x \in \mathbb{R}$*. If*

$$\sup_{x\in\mathbb{R}}[u_0'(x)] > \frac{1}{\gamma}\left(1 + \frac{1}{2} \cdot \sqrt{6 - \min\left\{-2,\ \gamma \cdot \inf_{x\in\mathbb{R}}[u_0'(x)]\right\}}\right), \tag{4.1.7}$$

then u_x *must blow up at some finite time.*

Regarding these results several remarks are in order.

i) Our threshold results in Theorems 4.1.2 and 4.1.3 are valid for any $0 < \gamma < \infty$. When the look-ahead distance $\gamma \to \infty$, both threshold conditions are reduced to $\sup_{x\in\mathbb{R}}[u_0'(x)] > 0$. On the other hand, when $\gamma \to \infty$, model (4.1.3) is reduced to the classical Lightwill-Whitham-Richards(LWR) model [Lighthill (1955); Richards (1956)],

$$\partial_t u + \partial_x(u(1-u)) = 0.$$

This local model can be verified to have finite time shock formation if initial data has positive slope $u_0' > 0$ at some point. Therefore, the threshold conditions identified are consistent with that of the LWR model.

ii) In a recent work [Li et al. (2011)] D. Li and T. Li presents several finite time shock formation scenarios of solutions to (4.1.3) with (4.1.4). Their approach is to analyze the solutions along two characteristic lines defined by $0 = u(t, X_1(t))$ and $1 = u(t, X_2(t))$, with which they justified that if there exist two points $\alpha_1 < \alpha_2$, such that $u_0(\alpha_1) = 0$ and $u_0(\alpha_2) = 1$, then u_x must blow up at some finite time. Compare to their result, our shock formation conditions in Theorems 4.1.2 and 4.1.3 may be viewed in the perspective of critical thresholds.

iii) The shock formation conditions in Theorem 4.1.2 and 4.1.3 are consistent with the numerical results obtained in [Kurganov et al. (2009)]. Indeed, a numerical comparison in [Kurganov et al. (2009)] of solutions to (4.1.3) with (4.1.4) for $\gamma = 0.1$ and $\gamma = 1$ indicates that the solution with $\gamma = 0.1$ remains smooth, while the solution with $\gamma = 1$ seems to contain a shock discontinuity.

iv) The threshold in (4.1.7) is bigger than that in (4.1.6). This observation suggests that under certain initial configuration, the traffic flow model with constant interaction potential may develop a congestion formation, while the model with the linear interaction potential may not. Roughly speaking, it is understood that the drivers with the linear potential are 'smarter' than the drivers with the constant potential.

v) For fixed $\gamma > 0$, both (4.1.6) and (4.1.7) reflect some balance between $\sup_{x\in\mathbb{R}}[u_0'(x)]$ and $\inf_{x\in\mathbb{R}}[u_0'(x)]$ for the finite time shock formation: if the non-positive term $\inf_{x\in\mathbb{R}}[u_0'(x)]$ is relatively small, then $\sup_{x\in\mathbb{R}}[u_0'(x)]$ needs to be large for the finite time shock formation. It indicates that not only the car density behind the traffic jam but also the car density ahead of the traffic jam contribute to the formation of congestion.

We now summarize the main arguments in our proofs to follow. For the proof of Theorem 4.1.1, we explore the classical energy method for hyperbolic problems, see e.g., [Dafermos (2005)]. Here we apply the Banach fixed-point theorem to the transformation S defined through $v = S(u)$, where v is solved from

$$\begin{cases} \partial_t v + F_u v_x + F_{\bar{u}} \bar{u}_x = 0, \\ v(t=0) = u_0. \end{cases} \tag{4.1.8}$$

We show that there exists $T > 0$ depending on initial data such that the mapping $v = S(u)$ exists and is a contraction. In so showing, detailed estimates of *non-local* terms are crucial, and allow us to track the dependence of T on the initial data.

For the proofs of Theorem 4.1.2-4.1.3, we trace the Lagrangian dynamics of $d := u_x$, which can be obtained from the Eulerian formulation:

$$(\partial_t + (1-2u)e^{-\bar{u}}\partial_x)d = e^{-\bar{u}}\left[2d^2 + 2(1-2u)\bar{u}_x d - u(1-u)\{\bar{u}_x\}^2 + u(1-u)\bar{u}_{xx}\right]. \tag{4.1.9}$$

The right hand side is quadratic in d, the a priori bound $0 \le u \le 1$ ensures the boundedness of both u and $\bar{u}_x$ involved in the coefficients. The key in our approach is to bound the non-local term $\bar{u}_{xx}$ in terms of

$$M(t) = \sup_{x\in\mathbb{R}}[u_x(x,t)] \text{ and } N(t) = \inf_{x\in\mathbb{R}}[u_x(x,t)]$$

attained at $x = \xi(t)$ and $x = \eta(t)$, respectively. This way we are able to obtain weakly coupled differential inequalities for both M and N, which yield the desired sub-thresholds.

This non-standard approach of tracing the dynamics d along two different curves originates in an idea of Seliger [Seliger (1968)] proving wave breaking for the Whitham equation. To carry out Seliger's formal analysis, one needs to assume that the curves $\xi(t)$ and $\eta(t)$ are smooth. This additional strong assumption was shown unnecessary later by Constantin and Escher [Constantin et al. (1998)]. In this work we are able to adapt these arguments to a class of nonlocal conservation laws (4.1.1).

From the proofs of Theorem 4.1.2-4.1.3 we observe that the one-sided interaction property of kernels (4.1.4) and (4.1.5) is crucial. Hence our threshold analysis for the traffic flow models is applicable to the class of nonlocal conservation laws (4.1.1) under the following assumptions:

(H_1). $F \in C^3(\mathbb{R}, \mathbb{R})$, and the kernel $K(r) \in W^{1,1}$ satisfying

$$K(r) = \begin{cases} Nondecreasing, & r \leq 0, \\ 0, & r > 0. \end{cases}$$

(H_2). $F(0, \cdot) = F(m, \cdot) = 0$ and

$$F_{uu} < 0, \;\; F_{\bar{u}\bar{u}} > 0, \quad F_{\bar{u}} < 0 \quad \text{for} \quad u \in [0, m].$$

The result can be stated as follows.

Theorem 4.1.4. *Consider* (4.1.1) *with* (5.1.2) *under assumptions* (H_1)*-*(H_2)*. If* $u_0 \in H^2$ *and* $0 \leq u_0(x) \leq m$ *for all* $x \in \mathbb{R}$*, then there exists a non-increasing function* $\lambda(\cdot)$ *such that if*

$$\sup_{x \in \mathbb{R}} [u_0'(x)] > \lambda(\inf_{x \in \mathbb{R}} [u_0'(x)]),$$

then u_x *must blow up at some finite time.*

We should point out that it was the threshold analysis for traffic flow models that led us to the thresholds (4.1.6), (4.1.7) in the first place, which in turn was then extended to the general class (4.1.1) as summarized in Theorem 5.1.1.

We now conclude this section by outlining the rest of the paper. In section 2, we prove local wellposedness for the class of nonlocal conservation laws (4.1.1). In section 3, we investigate sub-thresholds for nonlocal traffic flow models. We finally sketch the proof of Theorem 5.1.1 in the end of this paper.

4.2 Local Well-posedness and Regularity

In this section, we study the local well-posedness of (4.1.1). We consider a solution space as $u \in u_0(x) + B^T$, with $B^T := L^\infty([0,T]; H_x^2)$, which allows u to be non-zero at far field. By transformation

$$U = u - u_0,$$

we find the following equation for $U \in B^T$,

$$U_t + \partial_x F(U + u_0, \bar{U} + \bar{u}_0) = 0.$$

This lies in the same class as (4.1.1). With this in mind, from now on, we shall consider

$$u \in B^T := L^\infty([0,T]; H_x^2).$$

We prove the local wellposedness result by the fixed point argument. That is, we first define a transformation S as $v = S(u)$, where v is solved from the following equation

$$\begin{cases} \partial_t v + F_u v_x + F_{\bar{u}} \bar{u}_x = 0, \\ v(t=0) = u_0, \end{cases} \tag{4.2.1}$$

and then show this mapping has a fixed point.

We begin by verifying the existence of $v = S(u)$, which is carried out in a series of Lemmata 2.1-2.3. For simplicity, we take

$$a = F_u, \qquad b = -F_{\bar{u}} \bar{u}_x,$$

and then bound a and b in terms of u in the following lemma.

Lemma 4.2.1. *Suppose $u \in B^T$, $K \in W^{1,1}$. Then*

$$\|a_x\|_{H^1} \leq (k(1 + \|K\|_{L^1}))^2 (1 + \|u_x\|_\infty) \|u\|_{H^2} \tag{4.2.2}$$

and

$$\|b\|_{H^2} \leq k(1 + \|K\|_{L^1})^3 (1 + \|K_x\|_{L^1})(1 + \|u_x\|_\infty)^2 \|u\|_{H^2}, \tag{4.2.3}$$

where $k = k(F)$ is a constant depending on F. In particular, if $\sup_{t\in[0,T]} \|u\|_{H^2} \leq R$, then

$$\sup_{t\in[0,T]} \|a_x\|_{H^1} < c_a R^2 \quad \textit{and} \quad \sup_{t\in[0,T]} \|b\|_{H^2} < c_b R^3,$$

where $c_a = k(1+c_1)(1+\|K\|_{L^1})^2$, $c_b = k(1+c_1)^2(1+\|K\|_{W^{1,1}})^4$ *and* c_1 *is an embedding constant.*

Proof. We begin with some key inequalities for $\bar{u}$: using $\|w * K\|_{L^2} \leq \|K\|_{L^1}\|w\|_{L^2}$ and $K \in W^{1,1}$ we obtain

$$\begin{aligned}
\|\bar{u}_x\|_{L^2} &= \|K * u_x\|_{L^2} \leq \|K\|_{L^1}\|u_x\|_{L^2}, \\
\|\bar{u}_{xx}\|_{L^2} &= \|K * u_{xx}\|_{L^2} \leq \|K\|_{L^1}\|u_{xx}\|_{L^2}, \\
\|\bar{u}_{xxx}\|_{L^2} &= \|K_x * u_{xx}\|_{L^2} \leq \|K_x\|_{L^1}\|u_{xx}\|_{L^2}
\end{aligned} \tag{4.2.4}$$

and

$$\|\bar{u}_x\|_\infty \leq \|u_x\|_\infty \|K\|_{L^1}.$$

We calculate

$$\begin{aligned}
a_x &= F_{uu}u_x + F_{u\bar{u}}\bar{u}_x, \\
a_{xx} &= F_{uuu}u_x^2 + F_{uu\bar{u}}u_x\bar{u}_x + F_{uu}u_{xx} + F_{u\bar{u}u}u_x\bar{u}_x + F_{u\bar{u}\bar{u}}\bar{u}_x^2 + F_{u\bar{u}}\bar{u}_{xx},
\end{aligned}$$

so that

$$\|a_x\|_{L^2} \leq k\|u_x\|_{L^2} + k\|K\|_{L^1}\|u_x\|_{L^2} \leq k(1+\|K\|_{L^1})\|u\|_{H^2}.$$

$$\begin{aligned}
\|a_{xx}\|_{L^2} \leq k\Big(& \|u_x\|_\infty\|u_x\|_{L^2} + \|u_x\|_\infty\|K\|_{L^1}\|u_x\|_{L^2} + \|u_{xx}\|_{L^2} \\
& + \|u_x\|_\infty\|K\|_{L^1}\|u_x\|_{L^2} + \|u_x\|_\infty\|K\|_{L^1}^2\|u_x\|_{L^2} + \|K\|_{L^1}\|u_{xx}\|_{L^2}\Big) \\
\leq{} & k(1+\|u_x\|_\infty)(1+\|K\|_{L^1})^2\|u\|_{H^2}.
\end{aligned} \tag{4.2.5}$$

These together lead to (4.2.2).

We also calculate,

$$\begin{aligned}
b_x &= -F_{\bar{u}u}u_x\bar{u}_x - F_{\bar{u}\bar{u}}\bar{u}_x^2 - F_{\bar{u}}\bar{u}_{xx}, \\
b_{xx} &= -F_{\bar{u}uu}u_x^2\bar{u}_x - F_{\bar{u}u\bar{u}}u_x\bar{u}_x^2 - F_{\bar{u}u}u_x\bar{u}_x - F_{\bar{u}u}u_x\bar{u}_{xx} \\
&\quad - F_{\bar{u}\bar{u}u}u_x\bar{u}_x^2 - F_{\bar{u}\bar{u}\bar{u}}\bar{u}_x^3 - 2F_{\bar{u}\bar{u}}\bar{u}_x\bar{u}_{xx} \\
&\quad - F_{\bar{u}u}u_x\bar{u}_{xx} - F_{\bar{u}\bar{u}}\bar{u}_x\bar{u}_{xx} - F_{\bar{u}}\bar{u}_{xxx},
\end{aligned}$$

to obtain

$$\|b\|_{L^2} \leq k\|K\|_{L^1}\|u_x\|_{L^2}.$$

$$\begin{aligned}
\|b_x\|_{L^2} &\leq k\|u_x\|_\infty\|K\|_{L^1}\|u_x\|_{L^2} + k\|u_x\|_\infty\|K\|_{L^1}^2\|u_x\|_{L^2} + k\|K\|_{L^1}\|u_{xx}\|_{L^2} \\
&\leq k\Big((1+\|u_x\|_\infty)(1+\|K\|_{L^1})^2\Big)\|u\|_{H^2}.
\end{aligned}$$

$$\begin{aligned}
\|b_{xx}\|_{L^2} \leq k\Big(&\|u_x\|_\infty^2\|K\|_{L^1} + \|u_x\|_\infty^2\|K\|_{L^1}^2 + \|u_x\|_\infty\|K\|_{L^1} + \|u_x\|\|K\|_{L^1}\Big)\|u\|_{H^2} \\
&+ k\Big(\|u_x\|_\infty^2\|K\|_{L^1}^2 + \|u_x\|_\infty^2\|K\|_{L^1}^3 + 2\|u_x\|_\infty\|K\|_{L^1}^2\Big)\|u\|_{H^2} \\
&+ k\Big(\|u_x\|_\infty\|K\|_{L^1} + \|u_x\|_\infty\|K\|_{L^1}^2 + \|K_x\|_{L^1}\Big)\|u\|_{H^2}.
\end{aligned}$$

These estimates give (4.2.3). $\square$

Lemma 4.2.2 (A priori estimates)**.** *Suppose* $u \in B^T$. *A sufficiently smooth solution* v *of* (4.2.1) *must satisfy the energy estimates*

$$\sup_{t\in[0,T]}\|v(\cdot,t)\|_{L^2} \leq \Big(\|u_0\|_{L^2} + T\cdot\sup_{t\in[0,T]}\|b\|_{L^2}\Big)\exp\Big(\frac{1}{2}\int_0^T\|a_x\|_\infty d\tau\Big), \tag{4.2.6}$$

$$\sup_{t\in[0,T]}\|v(\cdot,t)\|_{H^2} \leq \Big(\|u_0\|_{H^2} + T\cdot\sup_{t\in[0,T]}\|b\|_{H^2}\Big)\exp\Big((\frac{3}{2}+c_1)\int_0^T\|a_x\|_{H^1}\,d\tau\Big), \tag{4.2.7}$$

where c_1 *is an embedding constant.*

Proof. Apply ∂_x^l to the first equation of (4.2.1) to obtain,

$$(\partial_x^l v)_t + a\cdot(\partial_x^l v)_x = h^l, \tag{4.2.8}$$

where $h^l = \partial_x^l b - \partial_x^l(av_x) + a(\partial_x^l v)_x$. Multiplying (4.2.8) by $\partial_x^l v$ and integrating over $\mathbb{R}$, we obtain,

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}(\partial_x^l v)^2\,dx = \int_{\mathbb{R}}a_x\frac{(\partial_x^l v)^2}{2} + \int_{\mathbb{R}}h^l\cdot(\partial_x^l v)\,dx. \tag{4.2.9}$$

This with $l=0$ leads to

$$\frac{d}{dt}\|v\|_{L^2}^2 = \int a_x v^2 dx + 2\int bv dx \leq \|a_x\|_\infty\|v\|_{L^2}^2 + 2\|b\|_{L^2}\|v\|_{L^2}.$$

That is,

$$\frac{d}{dt}\|v\|_{L^2} \leq \frac{1}{2}\|a_x\|_\infty\|v\|_{L^2} + \|b\|_{L^2}.$$

Upon integrating the above inequality, we obtain (4.2.6). Next, summing (4.2.9) for $l = 0, 1, 2$, we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|v\|_{H^2}^2 &= \frac{1}{2}\int_{\mathbb{R}} a_x \cdot \sum_{l=0}^{2}(\partial_x^l v)^2\,dx + \int_{\mathbb{R}}\sum_{l=0}^{2} h^l \cdot (\partial_x^l v)\,dx \\
&= \frac{1}{2}\int_{\mathbb{R}} a_x(v^2 - v_x^2 - 3v_{xx}^2)\,dx - \int_{\mathbb{R}} a_{xx}v_x v_{xx}\,dx \\
&\quad + \int_{\mathbb{R}}(bv + b_x v_x + b_{xx})v_{xx}\,dx \\
&\le \frac{3}{2}\|a_x\|_\infty\|v\|_{H^2}^2 + \|v_x\|_\infty\|a_{xx}\|_{L^2}\|v_{xx}\|_{L^2} + \|b\|_{H^2}\|v\|_{H^2} \\
&\le \left(\frac{3}{2} + c_1\right)\|a_x\|_{H^1}\|v\|_{H^2}^2 + \|b\|_{H^2}\|v\|_{H^2}.
\end{aligned} \tag{4.2.10}
$$

Therefore, we obtain

$$
\frac{d}{dt}\|v\|_{H^2} \le \left(\frac{3}{2} + c_1\right)\|a_x\|_{H^1}\|v\|_{H^2} + \|b\|_{H^2},
$$

which upon integration again gives (4.2.7). □

Lemma 4.2.3. *Suppose the initial data $v(x,0) = u_0 \in H^2$. Then for each $u \in B^T$, there exists a unique solution $v \in B^T$ of* (4.2.1).

Proof. Since $\sup_{t\in[0,T]}\|a_x\|_{H_x^1} < \infty$,

$$
\frac{dx}{dt} = a, \quad x(0) = x_0
$$

admits a unique solution $x = x(x_0, t)$ for each $x_0 \in \mathbb{R}$. Along $x(x_0,t)$, (4.2.1) reduces to

$$
\frac{dv}{dt} = b, \quad v(0) = u_0(x_0).
$$

Hence

$$
v(x(x_0,t),t) = u_0(x_0) + \int_0^t b(x(x_0,\tau),\tau)\,d\tau
$$

and the unique solution for (4.2.1) exists. □

Proof of Theorem 4.1.1: Let R be any number satisfying $R \ge 2\|u_0\|_{H^2}$, we define

$$
B_R^T := \left\{\omega \in L^\infty([0,T];H^2)\,|\,\omega(x,0) \equiv u_0,\ \sup_{t\in[0,T]}\|\omega(\cdot,t)\|_{H^2} \le R\right\}. \tag{4.2.11}
$$

Assume that $u \in B_R^T$, we then have

$$\|u(t)\|_\infty \leq c_0 R, \quad \|u_x(t)\|_\infty \leq c_1 R, \quad 0 \leq t \leq T,$$

where c_0 and c_1 are the embedding constants.

We first show that S maps B_R^T into B_R^T for some T small. From (4.2.7), it follows that

$$\begin{aligned}\sup_{t\in[0,T]} \|v(\cdot,t)\|_{H^2} &\leq \left(\frac{R}{2} + T\cdot c_b R^3\right)\exp\left(T\cdot(\frac{3}{2}+c_1)c_a R^2\right)\\ &\leq R,\end{aligned} \tag{4.2.12}$$

provided

$$T \leq T_1 := \frac{1}{3(2+c_1)(c_a+c_b)eR^2}.$$

Hence,

$$S : B_R^T \to B_R^T, \quad \forall T \leq T_1.$$

We next show that S is a contraction on B_R^T in the $L^\infty([0,T]; L_x^2)$ norm:

$$\sup_{t\in[0,T]} \|S(u_1) - S(u_2)\|_{L^2} \leq \frac{1}{2}\cdot \sup_{t\in[0,T]} \|u_1 - u_2\|_{L^2},\ \forall u_1, u_2 \in B_R^T. \tag{4.2.13}$$

Let $\tilde{v} := v_1 - v_2 = S(u_1) - S(u_2)$, then difference of (4.2.1) for v_2 and v_1, respectively, leads to

$$\tilde{v}_t + a(u_1)\tilde{v}_x = \tilde{b}, \quad \tilde{v}(0,x) = 0 \tag{4.2.14}$$

with

$$\tilde{b} =: -\{a(u_1) - a(u_2)\}v_{2x} + b(u_1) - b(u_2). \tag{4.2.15}$$

Applying (4.2.6) we have

$$\sup_{t\in[0,T]} \|\tilde{v}\|_{L^2} \leq T\cdot \sup_{t\in[0,T]} \|\tilde{b}(\cdot,t)\|_{L^2} \exp\left(\frac{1}{2}\int_0^T \|\partial_x a(u_1)\|_\infty \,d\tau\right). \tag{4.2.16}$$

In order to find a time interval such that the contraction property (4.2.13) holds, we need to estimate $\|\partial_x a(u_1)\|_\infty$ and $\|\tilde{b}(\cdot,t)\|_{L^2}$.

First we have

$$\begin{aligned}\|\partial_x a(u_1)\|_\infty &= \|F_{uu}u_{1x} + F_{u\bar{u}}\bar{u}_{1x}\|_\infty\\ &\leq k\big(\|u_{1x}\|_\infty + \|\bar{u}_{1x}\|_\infty\big)\\ &\leq k\big(c_1 R + c_1 R\|K\|_{L^1}\big)\\ &=: C_1 R.\end{aligned} \tag{4.2.17}$$

The first term in (4.2.15) is bounded as

$$\|\{a(u_1) - a(u_2)\}v_{2x}\|_{L^2} \leq C_1 R\|u_1 - u_2\|_{L^2}. \tag{4.2.18}$$

This can be seen from the following calculation:

$$\begin{aligned}
\|\{a(u_1) - a(u_2)\}v_{2x}\|_{L^2} &= \|\{F_u(u_1, \bar{u}_1) - F_u(u_2, \bar{u}_2)\}v_{2x}\|_{L^2} \\
&\leq \|\{F_u(u_1, \bar{u}_1) - F_u(u_2, \bar{u}_1)\}v_{2x}\|_{L^2} \\
&\quad + \|\{F_u(u_2, \bar{u}_1) - F_u(u_2, \bar{u}_2)\}v_{2x}\|_{L^2} \\
&\leq c_1 R k\bigg(\|u_1 - u_2\|_{L^2} + \|\bar{u}_1 - \bar{u}_2\|_{L^2}\bigg) \\
&\leq k c_1 R(1 + \|K\|_{L^1})\|u_1 - u_2\|_{L^2}.
\end{aligned}$$

If we assume $F_{\bar{u}}(0, \cdot) = 0$, then the last term in (4.2.15) has a similar bound:

$$\|b(u_1) - b(u_2)\|_{L^2} \leq C_2 R\|\tilde{u}\|_{L^2}. \tag{4.2.19}$$

To obtain this bound, we decompose it the following way

$$\begin{aligned}
b(u_1) - b(u_2) = -F_{\bar{u}}(u_1, \bar{u}_1)\{\bar{u}_{1x} - \bar{u}_{2x}\} - \bar{u}_{2x}\{F_{\bar{u}}(u_1, \bar{u}_1) - F_{\bar{u}}(u_2, \bar{u}_1)\} \\
- \bar{u}_{2x}\{F_{\bar{u}}(u_2, \bar{u}_1) - F_{\bar{u}}(u_2, \bar{u}_2)\}.
\end{aligned}$$

If we assume $F_{\bar{u}}(0, \cdot) = 0$, we have $F_{\bar{u}}(u_1, \bar{u}_1) = F_{\bar{u}u}(\xi, \bar{u}_1)u_1$,

$$\begin{aligned}
\|F_{\bar{u}}(u_1, \bar{u}_1)\{\bar{u}_{1x} - \bar{u}_{2x}\}\|_{L^2} &\leq k\|u_1\{\bar{u}_{1x} - \bar{u}_{2x}\}\|_{L^2} \\
&\leq k c_0 R\|\bar{u}_{1x} - \bar{u}_{2x}\|_{L^2} \\
&\leq k c_0 R\|K_x\|_{L^1}\|u_1 - u_2\|_{L^2}.
\end{aligned}$$

Applying the mean value property to the remaining terms gives that

$$\|b(u_1) - b(u_2)\|_{L^2} \leq k\{c_0\|K_x\|_{L^1} + c_1\|K\|_{L^1} + c_1\|K\|_{L^1}^2\}R\|u_1 - u_2\|_{L^2}.$$

Substituting (4.2.17), (4.2.18) and (4.2.19) into (4.2.16), we obtain

$$\sup_{t\in[0,T]} \|\tilde{v}\|_{L^2} \leq (C_1 + C_2)R \cdot T e^{C_1 R \cdot T} \sup_{t\in[0,T]} \|u_1 - u_2\|_{L^2}, \tag{4.2.20}$$

which ensures (4.2.13) if $T \leq T_2$ with

$$T_2 = \frac{1}{2e \cdot (C_1 + C_2)R}.$$

Therefore, for $0 < T < T^*$ with

$$T^* = \min\{T_1, T_2\} = \frac{1}{CR^2}\min\{1, R\},$$

the map S is a contraction on B_R^T in $L^\infty([0,T]; L_x^2)$ norm and thus possesses a unique fixed point u which is the unique solution of (4.1.1).

Note that without assuming $F_{\bar{u}}(0,\cdot) = 0$, a different bound than (4.2.19) is obtained

$$\|b(u_1) - b(u_2)\|_{L^2} \le (C_4 + C_3 R)\|\tilde{u}\|_{L^2},$$

hence T_2 satisfying

$$T_2 < \frac{1}{2e\{(C_1 + C_2)R + C_4\}}$$

still ensures the contraction. This ends the existence proof.

We prove the second part of Theorem 4.1.1 through the following corollary:

Corollary 4.2.4. *Let u be the solution obtained in Theorem 4.1.1 with a maximum life span $[0, T)$. Then*

$$\|u(t,\cdot)\|_{H^2} \le \|u_0\|_{H^2} \exp\left(k(1+c_1)(1+\|K\|_{W^{1,1}})^4 \int_0^t (1+\|u_x\|_\infty)^2 \, d\tau\right), \tag{4.2.21}$$

$0 \le t < T$, where c_1 is the Sobolev embedding constant. This infers that only one of the following occurs

i) $T = \infty$ and u is a global solution;

ii) $0 < T < \infty$ and

$$\lim_{t \to T-} \|\partial_x u(t,\cdot)\|_{L^\infty} = \infty.$$

Proof. We use again the estimate in (4.2.10), setting $v \equiv u$,

$$\frac{d}{dt}\|u\|_{H^2} \le \frac{3}{2}\|a_x\|_\infty \|u\|_{H^2} + \|u_x\|_\infty \|a_{xx}\|_{L^2} + \|b\|_{H^2}.$$

From $a_x = F_{uu}u_x + F_{u\bar{u}}\bar{u}_x$, it follows that $\|a_x\|_\infty \le k(1+\|K\|_{L^1})\|u_x\|_\infty$. Together with the estimates of $\|a_{xx}\|_{L^2}$ and $\|b\|_{H^2}$ in (4.2.5) and (4.2.3), respectively, we obtain

$$\frac{d}{dt}\|u\|_{H^2} \le k(1+c_1)(1+\|u_x\|_\infty)^2(1+\|K\|_{L^1})^3(1+\|K_x\|_{L^1})\|u\|_{H^2}.$$

Upon integration, we obtain (4.2.21). The claim in ii) follows from a contradiction argument: If $\lim_{t\to T-} \|u_x\|_\infty < \infty$, it would lead to the boundedness of $\|u\|_{H^2}$. One may therefore extend the solution for some $\tilde{T} > T$, which contradicts the assumption that $T < \infty$ is a maximal existence interval. $\square$

4.3 Sub-thresholds for Finite-time Shock Formation

4.3.1 Proof of the local existence theorem

In this subsection, we consider the traffic flow model with Arrhenius look-ahead dynamics:

$$\begin{cases} \partial_t u + \partial_x(u(1-u)e^{-\bar{u}}) = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{4.3.1}$$

where

$$\bar{u}(t,x) = \frac{1}{\gamma}\int_x^{x+\gamma} u(t,y)\,dy.$$

Here $\gamma > 0$ denotes look-ahead distance. In the theory of traffic flow, $u(t,x)$ represents a vehicle density normalized in the interval $[0,1]$.

We want identify some threshold condition for the shock formation of solutions to (4.3.1). From Corollary 4.2.4 we know that it suffices to track the dynamics of u_x. The idea is based on tracing

$$M(t) := \sup_{x\in\mathbb{R}}[u_x(x,t)] \text{ and } N(t) := \inf_{x\in\mathbb{R}}[u_x(x,t)].$$

The existence and differentiability (in almost everywhere sense) of $M(t)$ and $N(t)$ are proved in [Constantin et al. (1998)], which we summarize in the following.

Lemma 4.3.1. *(Theorem 2.1 in [Constantin et al. (1998)]) Let $T > 0$ and $u \in C^1([0,T];H^2)$. Then for every $t \in [0,T]$ there exists at least one point $\eta(t) \in \mathbb{R}$ with*

$$N(t) := \inf_{x\in\mathbb{R}}[u_x(t,x)] = u_x(t,\eta(t)),$$

and the function N is almost everywhere differentiable on $(0,T)$ with

$$\frac{dN}{dt}(t) = u_{tx}(t,\eta(t)) \quad a.e. \quad on\ (0,T).$$

We also state a useful result, which is proved in [Li et al. (2009)]. We then provide an extended version of it.

Lemma 4.3.2. *(Lemma 3.1. in [Li et al. (2009))] Consider the following quadratic equality for $A(t)$*

$$\frac{dA}{dt} = a(t)(A - b_1(t))(A - b_2(t)), \quad A(0) = A_0, \tag{4.3.2}$$

with $a(t) > 0$, $b_1(t) \leq b_2(t)$ and that $a(t)$, $b_1(t)$, $b_2(t)$ are uniformly bounded.

i) If $A_0 > \max b_2$, then $A(t)$ will experience a finite time blow-up.

ii) If there exists a constant $\bar{b}$ such that

$$b_1(t) \leq \bar{b} \leq b_2(t),$$

then (4.3.2) admits a unique global bounded solution satisfying

$$\min\{A_0, \min b_1\} \leq A(t) \leq \bar{b},$$

provided $A_0 \leq \bar{b}$.

With Lemma 4.3.2 we obtain the following:

Lemma 4.3.3. *Consider the following quadratic inequality,*

$$\frac{dB}{dt} \geq a(t)(B - b_1(t))(B - b_2(t)), \quad B(0) = B_0, \tag{4.3.3}$$

with $a(t) > 0$, $b_1(t) \leq b_2(t)$ and that $a(t)$, $b_1(t)$, $b_2(t)$ are uniformly bounded.

i) If $B_0 > \max b_2$, then $B(t)$ will experience a finite time blow-up.

ii) $\min\{B_0, \min b_1\} \leq B(t)$, *for $t \geq 0$ as long as $B(t)$ remains finite on the time interval* $[0, t]$.

Proof. i) Subtracting (4.3.2) from (4.3.3) gives

$$\frac{d}{dt}(B - A) \geq a(t)(B - A)(B + A - b_1 - b_2).$$

Integration leads to

$$(B - A)(t) \geq (B_0 - A_0) \exp\left(\int_0^t a(t)(B + A - b_1 - b_2)\, d\tau\right). \tag{4.3.4}$$

Therefore, $B_0 \geq A_0$ implies $B(t) \geq A(t)$. For any $B_0 > \max b_2$ set $A_0 = B_0$, then by Lemma 4.3.2, A_0 will lead to a finite time blow-up of $A(t)$. Hence, by (4.3.4), $B(t)$ will experience a finite time blow-up.

ii) Consider (4.3.2), it is easy to see that $\min\{A_0, \min b_1\} \leq A(t)$. Then (4.3.4) gives the result. $\square$

We remark that Lemma 4.3.3 remains valid even if the quadratic inequality holds almost everywhere.

We now turn to the proof of Theorem 4.1.2.

Proof of Theorem 4.1.2. Let $d := u_x$ and apply ∂_t to the first equation of (4.3.1),

$$\begin{aligned}\dot{d} &:= (\partial_t + (1-2u)e^{-\bar{u}}\partial_x)d \\ &= e^{-\bar{u}}\Big[2d^2 + 2(1-2u)\bar{u}_x d - u(1-u)\{\bar{u}_x\}^2 + u(1-u)\bar{u}_{xx}\Big].\end{aligned} \tag{4.3.5}$$

Define for $t \in [0, T)$,

$$\begin{aligned}M(t) &:= \sup_{x\in\mathbb{R}}[u_x(t,x)] = d(t, \xi(t)), \\ N(t) &:= \inf_{x\in\mathbb{R}}[u_x(t,x)] = d(t, \eta(t)).\end{aligned} \tag{4.3.6}$$

Then, along $(t, \xi(t))$, we have

$$\bar{u}_{xx} = \frac{1}{\gamma}\{u_x(\xi+\gamma) - u_x(\xi)\} \geq \frac{1}{\gamma}(-M+N),$$

and (4.3.5) can be written as,

$$\begin{aligned}\dot{M} &= e^{-\bar{u}}\Big(2M^2 + 2(1-2u)\bar{u}_x M - u(1-u)\{\bar{u}_x\}^2 + u(1-u)\bar{u}_{xx}\Big) \quad a.e. \\ &\geq e^{-\bar{u}}\Big(2M^2 + 2(1-2u)\bar{u}_x M - u(1-u)\{\bar{u}_x\}^2 + u(1-u)\frac{(-M+N)}{\gamma}\Big).\end{aligned} \tag{4.3.7}$$

Along $(t, \eta(t))$, we have

$$\bar{u}_{xx} = \frac{1}{\gamma}\{u_x(\eta+\gamma) - u_x(\eta)\} \geq 0,$$

and (4.3.5) can be written as,

$$\begin{aligned}\dot{N} &= e^{-\bar{u}}\Big(2N^2 + 2(1-2u)\bar{u}_x N - u(1-u)\{\bar{u}_x\}^2 + u(1-u)\bar{u}_{xx}\Big) \quad a.e. \\ &\geq e^{-\bar{u}}\Big(2N^2 + 2(1-2u)\bar{u}_x N - u(1-u)\{\bar{u}_x\}^2\Big).\end{aligned} \tag{4.3.8}$$

(4.3.8) can be written as

$$\dot{N} \geq 2e^{-\bar{u}}(N - N_1)(N - N_2) \quad a.e.\ , \tag{4.3.9}$$

where

$$N_1(u, \bar{u}_x) = \frac{-(1-2u)\bar{u}_x - \sqrt{\{(1-2u)\bar{u}_x\}^2 + 2u(1-u)\bar{u}_x^2}}{2}$$

and

$$N_2(u, \bar{u}_x) = \frac{-(1-2u)\bar{u}_x + \sqrt{\{(1-2u)\bar{u}_x\}^2 + 2u(1-u)\bar{u}_x^2}}{2}.$$

We note that $N_1 \leq 0 \leq N_2$ because $0 \leq u(t) \leq 1$. It can be shown later that N_1 is uniformly bounded from below,

$$N_1 \geq -\frac{1}{\gamma}. \tag{4.3.10}$$

Applying Lemma 4.3.3 (ii) to (4.3.9) with $\min_{0 \leq u \leq 1,\ |\omega| \leq \frac{1}{\gamma}} N_1(u, \omega) = -\frac{1}{\gamma}$, we obtain

$$N(t) \geq \min\left\{ -\frac{1}{\gamma},\ N(0) \right\} =: \frac{\tilde{N}_0}{\gamma}.$$

Substituting this lower bound into (4.3.7), we obtain

$$\dot{M} \geq e^{-\bar{u}}\left(2M^2 + \left\{ 2(1-2u)\bar{u}_x - \frac{u(1-u)}{\gamma} \right\} M - u(1-u)\bar{u}_x^2 + \frac{u(1-u)\tilde{N}_0}{\gamma^2} \right) \ a.e.$$

Rewriting of this inequality gives

$$\dot{M} \geq 2e^{-\bar{u}}(M - M_1)(M - M_2) \quad a.e.\ , \tag{4.3.11}$$

where $M_2 (\geq M_1)$ is given by

$$M_2 := \frac{-\{2(1-2u)\bar{u}_x - \frac{u(1-u)}{\gamma}\} + \sqrt{\{2(1-2u)\bar{u}_x - \frac{u(1-u)}{\gamma}\}^2 + 8u(1-u)\bar{u}_x^2 - 8\frac{u(1-u)\tilde{N}_0}{\gamma^2}}}{4}.$$

We claim that M_2 has an uniform upper bound,

$$M_2 \leq \frac{1}{\gamma}\left[\frac{1}{2} + \frac{\sqrt{2}}{4} \cdot \sqrt{3 - \tilde{N}_0} \right]. \tag{4.3.12}$$

By Lemma 4.3.3 (i), if

$$M(0) > \frac{1}{\gamma}\left[\frac{1}{2} + \frac{\sqrt{2}}{4} \cdot \sqrt{3 - \tilde{N}_0} \right],$$

then $M(t)$ will blow up a finite time. This is exactly the threshold condition as stated in Theorem 4.1.2.

To complete our proof we still need to verify both claims (4.3.12) and (4.3.10).

To verify (4.3.12), we set

$$v := \gamma \cdot \bar{u}_x = u(x+\gamma) - u(x).$$

From $0 \leq u(t) \leq 1$ it follows that $-1 \leq v \leq 1$. If suffices to find upper bound for M_2 over the set

$$\Omega := \{(u,v) \in \mathbb{R}^2 \mid 0 \leq u \leq 1, \quad -1 \leq v \leq 1\}.$$

In fact,

$$\begin{aligned} M_2 &= \frac{-\{2(1-2u)v - u(1-u)\} + \sqrt{\{2(1-2u)v - u(1-u)\}^2 + 8u(1-u)(v^2 - \tilde{N}_0)}}{4\gamma} \\ &\leq \frac{1}{4\gamma}\big[2 + \sqrt{4 + 2(1-\tilde{N}_0)}\big]. \end{aligned}$$

Here, we use $\max_{(u,v)\in\Omega}\{-2(1-2u)v + u(1-u)\} = 2$ which can be verified easily since the underlying function is linear in v and quadratic in u. For the next one, $\max_{(u,v)\in\Omega}\{8u(1-u)(v^2 - \tilde{N}_0)\} = 2(1-\tilde{N}_0)$ is used, which is obtained from the upper bound $u(1-u) \leq 1/4$.

Finally, we are left with the verification of (4.3.10). With v defined above, we have

$$Q := \gamma N_1 = \frac{-(1-2u)v - \sqrt{\{(1-2u)v\}^2 + 2u(1-u)v^2}}{2}.$$

By rearranging,

$$\begin{aligned} Q^2 &= \frac{u(1-u)v^2}{2} - Q \cdot (1-2u)v \\ &\leq \frac{u(1-u)v^2}{2} + \epsilon Q^2 + \frac{(1-2u)^2}{4\epsilon}v^2, \quad 0 < \epsilon < 1. \end{aligned} \tag{4.3.13}$$

It follows that

$$\begin{aligned} (1-\epsilon)Q^2 &\leq \frac{v^2}{4\epsilon}\{(1-2u)^2 + 2\epsilon u(1-u)\} \\ &\leq \frac{1}{4\epsilon}, \end{aligned} \tag{4.3.14}$$

where the maximum value is achieved at $\partial\Omega$. This gives

$$Q^2 \leq \frac{1}{4\epsilon(1-\epsilon)}.$$

Since ϵ is arbitrary, we choose $\epsilon = \frac{1}{2}$ to get $Q^2 \leq 1$, hence $Q \geq -1$, which gives (4.3.10).

4.3.2 Proof of shock formation-constant kernel case

We rewrite the traffic flow model (4.1.3) with the linear potential as

$$\partial_t u + \partial_x(u(1-u)e^{-\tilde{u}}) = 0, \tag{4.3.15}$$

where

$$\tilde{u}(t,x) = \frac{2}{\gamma}\int_x^{x+\gamma} \big(1 + \frac{x-y}{\gamma}\big)u(t,y)\,dy. \tag{4.3.16}$$

Let $d := u_x$ and apply ∂_x to (4.3.15),

$$\begin{aligned}\dot{d} &= (\partial_t + (1-2u)e^{-\tilde{u}}\partial_x)d \\ &= e^{-\tilde{u}}\Big[2d^2 + 2(1-2u)\tilde{u}_x d - u(1-u)\{\tilde{u}_x\}^2 + u(1-u)\tilde{u}_{xx}\Big].\end{aligned} \tag{4.3.17}$$

Here,

$$\begin{aligned}\tilde{u}_x &= -\frac{2}{\gamma}\Big\{u(x) - \frac{1}{\gamma}\int_x^{x+\gamma} u(y)\,dy\Big\} = -\frac{2}{\gamma}(u-\bar{u}), \\ \tilde{u}_{xx} &= -\frac{2}{\gamma}(u_x - \bar{u}_x),\end{aligned} \tag{4.3.18}$$

where

$$\bar{u} = \frac{1}{\gamma}\int_x^{x+\gamma} u(y)\,dy$$

as defined in the previous section. Define for $t \in [0,T)$,

$$\begin{aligned}M(t) &:= \sup_{x\in\mathbb{R}}[u_x(t,x)] = d(t,\xi(t)), \\ N(t) &:= \inf_{x\in\mathbb{R}}[u_x(t,x)] = d(t,\eta(t)).\end{aligned} \tag{4.3.19}$$

The existence of $\xi(t)$ and $\eta(t)$ is justified by Theorem 2.1 in [Constantin et al. (1998)]. Then, along $(t,\xi(t))$, (4.3.17) can be written as,

$$\begin{aligned}\dot{M} &= e^{-\tilde{u}}\Big(2M^2 + 2(1-2u)\tilde{u}_x M - u(1-u)\{\tilde{u}_x\}^2 + u(1-u)\tilde{u}_{xx}\Big)\ a.e. \\ &\geq e^{-\tilde{u}}\Big(2M^2 + 2(1-2u)\tilde{u}_x M - u(1-u)\{\tilde{u}_x\}^2 + u(1-u)\frac{2(N-M)}{\gamma}\Big),\end{aligned} \tag{4.3.20}$$

where the last inequality follows from the fact that

$$\tilde{u}_{xx}(t,\xi) = \frac{2}{\gamma}(\bar{u}_x - M) \geq \frac{2}{\gamma}(N-M).$$

And along $(t, \eta(t))$, (4.3.17) can be written as,

$$\begin{aligned}\dot{N} &= e^{-\tilde{u}}\Big(2N^2 + 2(1-2u)\tilde{u}_x N - u(1-u)\{\tilde{u}_x\}^2 + u(1-u)\tilde{u}_{xx}\Big)\ a.e.\\ &\geq e^{-\tilde{u}}\Big(2N^2 + 2(1-2u)\tilde{u}_x N - u(1-u)\{\tilde{u}_x\}^2\Big),\end{aligned} \tag{4.3.21}$$

where the last inequality follows from the fact that

$$\tilde{u}_{xx}(t,\eta) = \frac{2}{\gamma}(\bar{u}_x - N) \geq 0.$$

(4.3.21) can be written as

$$\dot{N} \geq 2e^{-\tilde{u}}(N - N_1)(N - N_2) \quad a.e.\ , \tag{4.3.22}$$

where

$$N_1 = \frac{-(1-2u)\tilde{u}_x - \sqrt{\{(1-2u)\tilde{u}_x\}^2 + 2u(1-u)\tilde{u}_x^2}}{2}$$

and

$$N_2 = \frac{-(1-2u)\tilde{u}_x + \sqrt{\{(1-2u)\tilde{u}_x\}^2 + 2u(1-u)\tilde{u}_x^2}}{2}.$$

We note that

$$N_1 \leq 0 \leq N_2$$

because $0 \leq u(t) \leq 1$.

By using the fact that $0 \leq u \leq 1$, and $-2 \leq \gamma\tilde{u}_x \leq 2$, it can be shown that N_1 is uniformly bounded from below,

$$N_1 \geq -\frac{2}{\gamma}.$$

The verification of this inequality is similar to the one in the proof (4.3.10), details are omitted. With the lower bound of $N_1(t)$, Lemma 4.3.3 (ii) when applied to (4.3.22) gives

$$N(t) \geq \min\Big\{-\frac{2}{\gamma},\ N(0)\Big\} =: \frac{\tilde{N}_0}{\gamma}. \tag{4.3.23}$$

Substituting this lower bound into (4.3.20), we obtain

$$\begin{aligned}\dot{M} &\geq e^{-\tilde{u}}\Big[2M^2 + \Big\{2(1-2u)\tilde{u}_x - \frac{2u(1-u)}{\gamma}\Big\}M - u(1-u)\tilde{u}_x^2 + \frac{2u(1-u)\tilde{N}_0}{\gamma^2}\Big]\\ &= 2e^{-\tilde{u}}(M - M_1)(M - M_2) \quad a.e.\end{aligned} \tag{4.3.24}$$

In order to apply Lemma 4.3.3 (i) to (4.3.24), we proceed to find the upper bound of $M_2(\geq M_1)$. Let $v := \gamma \cdot \tilde{u}_x = -2(u-\bar{u})$, then from the fact that $0 \leq u, \bar{u} \leq 1$, we know that $-2 \leq v \leq 2$. We also let

$$\Omega := \{(u,v) \in \mathbb{R}^2 \mid 0 \leq u \leq 1, \quad -2 \leq v \leq 2\}$$

then M_2 and it's upper bound are given by

$$\begin{aligned} M_2 &= \frac{-\{2(1-2u)v - 2u(1-u)\} + \sqrt{\{2(1-2u)v-2u(1-u)\}^2 + 8u(1-u)(v^2-2\tilde{N}_0)}}{4\gamma} \\ &\leq \frac{1}{4\gamma}\left[4 + \sqrt{16 + 2(4-2\tilde{N}_0)}\right]. \end{aligned} \tag{4.3.25}$$

Here, we use $\max_{(u,v)\in\Omega}\{-2(1-2u)v + u(1-u)\} = 4$ which can be verified easily since the underlying function is linear in v and quadratic in u. We also use $u(1-u) \leq \frac{1}{4}$ in bounding the term $8u(1-u)(v^2 - 2\tilde{N}_0)$. Therefore, by Lemma 4.3.3 (i), if

$$M(0) > \frac{1}{\gamma}\left[1 + \frac{1}{2}\cdot\sqrt{6-\tilde{N}_0}\right],$$

then $M(t)$ experience a finite time blow up. Hence we obtain the desired result.

4.3.3 Proof of shock formation-linear kernel case

We only sketch the proof since it is entirely similar to that in the previous sections. Let $d := u_x$ and apply ∂_x to the first equation of (4.1.1) to obtain

$$(\partial_t + F_u \cdot \partial_x)d = -F_{uu}d^2 - 2F_{u\bar{u}}\bar{u}_x d - F_{\bar{u}\bar{u}}\bar{u}_x^2 - F_{\bar{u}}\bar{u}_{xx}. \tag{4.3.26}$$

It can be shown that $0 \leq u \leq m$, and therefore

$$|\bar{u}| \leq m\|K\|_{W^{1,1}}, \quad |\bar{u}_x| \leq m\|K\|_{W^{1,1}}.$$

To find the bound of $\bar{u}_{xx}$, we define for $t \in [0,T)$,

$$\begin{aligned} M(t) &:= \sup_{x\in\mathbb{R}}[u_x(t,x)] = d(t,\xi(t)), \\ N(t) &:= \inf_{x\in\mathbb{R}}[u_x(t,x)] = d(t,\eta(t)). \end{aligned} \tag{4.3.27}$$

From (5.1.2), it follows that

$$\bar{u}_{xx}(t,x) = \int_{-\infty}^{0} K'(z)u_x(t,x-z)\,dz - K(0)u_x(t,x).$$

Therefore, along $\xi(t)$,

$$K(0)(N-M) \leq \bar{u}_{xx} \leq 0,$$

and (4.3.26) is reduced to

$$\dot{M} \geq -F_{uu}M^2 - 2F_{u\bar{u}}\bar{u}_x M - F_{\bar{u}\bar{u}}\bar{u}_x^2 - F_{\bar{u}}K(0)(N-M) \quad a.e.\ . \tag{4.3.28}$$

Also, along $\eta(t)$,

$$0 \leq \bar{u}_{xx} \leq K(0)(M-N).$$

and (4.3.26) is reduced to

$$\dot{N} \geq -F_{uu}N^2 - 2F_{u\bar{u}}\bar{u}_x N - F_{\bar{u}\bar{u}}\bar{u}_x^2 = -F_{uu}(N-N_1)(N-N_2) \quad a.e.\ , \tag{4.3.29}$$

where

$$N_1(u,\bar{u}_x) = \frac{F_{u\bar{u}}\bar{u}_x - \sqrt{\left(F_{u\bar{u}}^2 - F_{uu}F_{\bar{u}\bar{u}}\right)\bar{u}_x^2}}{-F_{uu}}.$$

From (4.3.29) we infer the lower bound of $N(t)$ as

$$N(t) \geq \min\{N(0), \min_{0\leq u\leq m, |v|\leq m\|K\|_{W^{1,1}}} N_1(u,v)\} =: \tilde{N}_0,$$

Substituting this lower bound into (4.3.28), we obtain

$$\begin{aligned}\dot{M} &\geq -F_{uu}M^2 - 2F_{u\bar{u}}\bar{u}_x M - F_{\bar{u}\bar{u}}\bar{u}_x^2 - F_{\bar{u}}K(0)(\tilde{N}_0 - M) \\ &= -F_{uu}(M-M_1)(M-M_2) \quad a.e.,\end{aligned}$$

where

$$M_2(u,\bar{u}_x) = \frac{2F_{u\bar{u}}\bar{u}_x - F_{\bar{u}}K(0) + \sqrt{\{2F_{u\bar{u}}\bar{u}_x - F_{\bar{u}}K(0)\}^2 - 4\{F_{uu}F_{\bar{u}\bar{u}}\bar{u}_x^2 + F_{uu}F_{\bar{u}}K(0)\tilde{N}_0\}}}{-2F_{uu}}.$$

Therefore, by Lemma 4.3.3 (i), if

$$M(0) > \max_{0\leq u\leq m, |v|\leq m\|K\|_{W^{1,1}}} M_2(u,v) =: \lambda(N(0)),$$

then $M(t)$ will blow up in finite time. Hence we obtain the desired result.

CHAPTER 5. WELL-POSEDNESS OF THE GLOBAL ENTROPY SOLUTION TO A CLASS OF NONLOCAL CONSERVATION LAWS

Yongki Lee and Hailiang Liu

Abstract We investigate a class of nonlocal conservation laws with the nonlinear advection coupling both local and nonlocal mechanisms, which arise in several applications such as traffic flows and the collective motion of cells. We first introduce the Kružkov type entropy solution. Next, by adapting the doubling of variables method and the method of vanishing viscosity we present a uniqueness and existence result within the class of entropy solutions for the initial value problem.

5.1 Introduction

We investigate a class of nonlocal conservation laws,

$$\begin{cases} \partial_t u + \partial_x F(u, \bar{u}) = 0, & t > 0,\ x \in \mathbb{R}, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{5.1.1}$$

where u is the unknown, F is a given smooth function, and $\bar{u}$ is given by

$$\bar{u}(t,x) = (K * u)(t,x) = \int_{\mathbb{R}} K(x-y)u(t,y)\,dy, \tag{5.1.2}$$

where K is assumed in $W^{1,1}(\mathbb{R}) \cap L^\infty(\mathbb{R})$. The advection couples both local and nonlocal mechanism. This class of conservation laws appears in several applications, including traffic flows [Kurganov et al. (2009); Sopasakis et al. (2006)], the collective motion of biological cells [Dolak et al. (2005); Burger et al. (2008); Perthame et al. (2009)], dispersive water waves [Whitham (1974); Holm et al. (2005); Degasperis et al. (1999); Liu (2006)], the radiating gas motion [Hamer (1971); Rosenau (1989); Liu et al. (2001)], high-frequency waves in relaxing medium [Hunter (1990); Parkes (2002); Vakhnenko (1992)], and the kinematic sedimentation [Kynch (1952); Zumbrun (1999); Karlsen et al. (2011)].

Our interests in the class of nonlocal conservation laws are two fold:

i) The criterion for the propagation of C^1 solution regularity. As is known, the typical well-posedness result asserts that either a solution of a time-dependent PDE exists for all time or else there is a finite time such that some norm of the solution becomes unbounded as the life span is approached. The natural question is whether there is a critical threshold for the initial data such that the persistence of the C^1 solution regularity depends only on crossing such a critical threshold. This concept of critical threshold(CT) and associated methodology was originated and developed in a series of papers by Engelberg, Liu and Tadmor [Engelberg et al. (2001); Liu et al. (2002, 2003)] for a class of Euler-Poisson equations. Following their CT concept, we identified sub-thresholds for finite time shock formation in a class of nonlocal conservation laws. Here we revisit the authors' finite time shock formation condition in [Lee et al. (2014)]. To carry out the finite time shock formation analysis, two assumptions are made:
(H_1). $F \in C^3(\mathbb{R}, \mathbb{R})$, and the kernel $K(r) \in W^{1,1}$ satisfying

$$K(r) = \begin{cases} Nondecreasing, & r \leq 0, \\ 0, & r > 0. \end{cases}$$

(H_2). $F(0, \cdot) = F(m, \cdot) = 0$ and

$$F_{uu} < 0, \ F_{\bar{u}\bar{u}} > 0, \quad F_{\bar{u}} < 0 \quad \text{for} \quad u \in [0, m].$$

The result can be stated as follows.

Theorem 5.1.1. *Consider* (5.1.1) *with* (5.1.2) *under assumptions* (H_1)*-*(H_2)*. If* $u_0 \in H^2$ *and* $0 \leq u_0(x) \leq m$ *for all* $x \in \mathbb{R}$*, then there exists a non-increasing function* $\lambda(\cdot)$ *such that if*

$$\sup_{x \in \mathbb{R}} [u_0'(x)] > \lambda(\inf_{x \in \mathbb{R}} [u_0'(x)]),$$

then u_x *must blow up at some finite time.*

We should point out that it was the threshold analysis for traffic flow models with Arrhenius look-ahead dynamics in the first place, which in turn was then extended to the general class (5.1.1) as summarized in the above theorem.

ii) The global well-posedness of problem (5.1.1). In particular, the main results of this paper are the uniqueness and existence of entropy solutions. It is well known that the initial value problem for a scalar conservation law may admit more than one weak solution, so a selection criterion must be imposed in order to single out the physically relevant solution. In this paper, this is done by defining the Kružkov-type [Kružkov (1970)] entropy solution and proving existence and uniqueness for this solution. In this introduction section we define an entropy solution by using the Kružkov-type entropy inequality. To facilitate notation we define $S_T = \mathbb{R} \times (0, T)$. We also let

$$F \in C^2(\mathbb{R} \times \mathbb{R}) \textit{ and } F(0, \cdot) = F(m, \cdot) = 0.$$

Definition 5.1.2. *A measurable function u is an entropy solution of the initial value problem* (5.1.1) *if it satisfies the following three conditions:*

1)

$$u \in L^\infty(S_T) \cap L^1(S_T) \cap BV(S_T),$$

where the bounded variation is defined as usual

$$|u(\cdot, t)|_{BV(\mathbb{R})} := \int_{\mathbb{R}} |u_x(x, t)|\, dx.$$

2)

$$\lim_{t \to 0} \int_{\mathbb{R}} |u(x, t) - u_0(x)|\, dx = 0.$$

3) For all nonnegative test functions $\psi \in C_0^\infty(S_T)$,

$$\iint_{S_T} \partial_t \psi \cdot |u - k| + sgn(u - k)\big[F(u, \bar{u}) - F(k, \bar{u})\big]\psi_x - sgn(u - k)F(k, \bar{u})_x \psi \, dxdt \geq 0, \quad (5.1.3)$$

$\forall k \in \mathbb{R}$.

With this definition of entropy solutions, the main result is stated as follows:

Theorem 5.1.3. *Assume that u and v are entropy solutions of problem* (5.1.1) *with initial data u_0 and v_0, respectively. Then, for any $T > 0$ there exists s constant C such that*

$$\|u(\cdot, t) - v(\cdot, t)\|_{L^1(\mathbb{R})} \leq C \|u_0 - v_0\|_{L^1(\mathbb{R})} \qquad \forall t \in [0, T]. \quad (5.1.4)$$

In particular, an entropy solution of (5.1.1) *is unique.*

It should be pointed out that the proof is based on the method of doubling of variables introduced by Kružkov [Kružkov (1970)], and many technical details in the proof are motivated by the previous work of [Karlsen et al. (2011)]. In which, Karlsen et. al. studied the nonlocal conservation laws of the form,

$$\partial_t u + \partial_x (u(1-u)^\alpha V(K_a * u)) = 0, \quad x \in \mathbb{R}, \quad t > 0,$$

together with the initial data

$$u(x,0) = u_0(x), \quad 0 \le u_0(x) \le 1, \quad x \in \mathbb{R}.$$

Here, K_a is a symmetric and nonnegative function with support on $[-2a, 2a]$ and

$$\int_{\mathbb{R}} K_a(x)\, dx = 1.$$

Their main results are the uniqueness and existence of entropy solutions. This is done by proving convergence of a difference-quadrature scheme based on the standard Lax-Friedrichs discretization. We should point out that the advection term and kernel K in (5.1.1) are slightly generalized, provided that α is non-zero.

We now conclude this section by outlining the rest of the chapter. In section 2, by adapting the doubling of variables method, we prove a uniqueness result within the class of entropy solutions for the initial value problem. In section 3, we prove an existence result using the method of vanishing viscosity.

5.2 Proof of Theorem 5.1.3

Consider nonnegative Lipschitz continuous function $\phi(x,t,\bar{x},\bar{t})$ having compact support in its arguments. Fix $(\bar{x},\bar{t})$, and let $\psi(x,t) = \phi(x,t,\bar{x},\bar{t})$, $v = v(\bar{x},\bar{t})$ and $u = u(x,t)$ in (5.1.3) to obtain

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}} \Big\{ \underbrace{\partial_t \phi(x,t,\bar{x},\bar{t})|u-v|}_{I_0} + \underbrace{sgn(u-v)\big[F(u,\bar{u}) - F(v,\bar{u})\big]\partial_x \phi(x,t,\bar{x},\bar{t}) - sgn(u-v)F(v,\bar{u})_x \phi}_{I_1} \Big\}\, dxdt \ge 0. \tag{5.2.1}$$

Here and below we use $\mathbb{R}\times\mathbb{R}^+$ instead of S_T, since ϕ has compact support. Interchanging the roles of u and v, for any fixed (x,t):

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}}\Big\{\underbrace{\partial_{\bar t}\phi(x,t,\bar x,\bar t)|v-u|}_{J_0} \\ +\underbrace{sgn(v-u)\big[F(v,\bar v)-F(u,\bar v)\big]\partial_{\bar x}\phi(x,t,\bar x,\bar t)-sgn(v-u)F(u,\bar v)_{\bar x}\phi}_{J_1}\Big\}\,d\bar x d\bar t\geq 0. \tag{5.2.2}$$

Note that we can write

$$\begin{aligned} I_1 &= sgn(u-v)\big[F(u,\bar u)-F(v,\bar u)\big]\partial_x\phi(x,t,\bar x,\bar t)-sgn(u-v)F(v,\bar u)_x\phi \\ &= \underbrace{sgn(u-v)\Big(F(u,\bar u)-F(v,\bar v)\Big)\partial_x\phi}_{I_{1,1}}+\underbrace{sgn(u-v)\Big[\big(F(v,\bar v)-F(v,\bar u)\big)\phi\Big]_x}_{I_{1,2}} \end{aligned} \tag{5.2.3}$$

$$\begin{aligned} J_1 &= sgn(v-u)\big[F(v,\bar v)-F(u,\bar v)\big]\partial_{\bar x}\phi(x,t,\bar x,\bar t)-sgn(v-u)F(u,\bar v)_{\bar x}\phi \\ &= \underbrace{sgn(v-u)\Big(F(v,\bar v)-F(u,\bar u)\Big)\partial_{\bar x}\phi}_{J_{1,1}}+\underbrace{sgn(v-u)\Big[\big(F(u,\bar u)-F(u,\bar v)\big)\phi\Big]_{\bar x}}_{J_{1,2}} \end{aligned} \tag{5.2.4}$$

Integrating over $\mathbb{R}\times[0,\infty)$ (5.2.1), with respect to $(\bar x,\bar t)$, and (5.2.2), with respect to (x,t), and then adding the resulting inequalities yields

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}}\int_{\mathbb{R}^+}\int_{\mathbb{R}}|u-v|(\partial_t+\partial_{\bar t})\phi+(I_{1,1}+J_{1,1})+(I_{1,2}+J_{1,2})\,dxdtd\bar xd\bar t\geq 0. \tag{5.2.5}$$

Consider any non-negative Lipschitz function ψ on $\mathbb{R}\times[0,\infty)$, with compact support and a smooth non-negative function ρ with compact support and

$$\int_{-\infty}^{\infty}\rho(x)\,dx=\frac{1}{2}.$$

For small $\epsilon>0$, we shall proceed with (5.2.5) using the test function

$$\phi(x,t,\bar x,\bar t)=\epsilon^{-2}\psi\Big(\frac{x+\bar x}{2},\frac{t+\bar t}{2}\Big)\rho\Big(\frac{t-\bar t}{2\epsilon}\Big)\rho\Big(\frac{x-\bar x}{2\epsilon}\Big), \tag{5.2.6}$$

and then let $\epsilon\to 0$ in order to obtain (5.1.4).

We first consider the first integrand in (5.2.5).

$$
\begin{aligned}
&\int_{\mathbb{R}^+}\int_{\mathbb{R}}\int_{\mathbb{R}^+}\int_{\mathbb{R}} |u(x,t)-v(\bar{x},\bar{t})|(\partial_t+\partial_{\bar{t}})\phi(x,t,\bar{x},\bar{t})\,dxdtd\bar{x}d\bar{t} \\
&= \int_{\mathbb{R}^+}\int_{\mathbb{R}}\int_{\mathbb{R}^+}\int_{\mathbb{R}} |u(x,t)-v(\bar{x},\bar{t})|\epsilon^{-2}\psi_t\Big(\frac{x+\bar{x}}{2},\frac{t+\bar{t}}{2}\Big)\rho\Big(\frac{t-\bar{t}}{2\epsilon}\Big)\rho\Big(\frac{x-\bar{x}}{2\epsilon}\Big)\,dxdtd\bar{x}d\bar{t} \\
&= \int_{\mathbb{R}^+}\int_{\mathbb{R}}\int_{\mathbb{R}^+}\int_{\mathbb{R}} \Big\{|u(x,t)-v(\bar{x},\bar{t})|-|u(x,t)-v(x,t)|\Big\}\epsilon^{-2}\psi_t(\cdot,\cdot)\rho(\,\cdot\,)\rho(\,\cdot\,)\,dxdtd\bar{x}d\bar{t} \\
&+ \int_{\mathbb{R}^+}\int_{\mathbb{R}}\int_{\mathbb{R}^+}\int_{\mathbb{R}} |u(x,t)-v(x,t)|\epsilon^{-2}\psi_t(\cdot,\cdot)\rho(\,\cdot\,)\rho(\,\cdot\,)\,dxdtd\bar{x}d\bar{t} \\
&:= L_1+L_2.
\end{aligned}
$$

We find that

$$
|L_1| \le \frac{C}{\epsilon^2}\int_{\mathbb{R}^+}\int_{\mathbb{R}}\int_{\mathbb{R}^+}\int_{\mathbb{R}} |v(x,t)-v(\bar{x},\bar{t})|\rho\Big(\frac{t-\bar{t}}{2\epsilon}\Big)\rho\Big(\frac{x-\bar{x}}{2\epsilon}\Big)\,dxdtd\bar{x}d\bar{t},
$$

where the constant C does not depend on ϵ. By Lemma 2 in [Kružkov (1970)], we have

$$
\lim_{\epsilon\to 0}|L_1| = 0.
$$

Consider the integral L_2, substituting $x=\eta$, $t=\alpha$, $(t-\bar{t})/2\epsilon=\beta$, $(x-\bar{x})/2\epsilon=\xi$ and taking into account the fact that Jacobian of transform $dxdtd\bar{x}d\bar{t}\to d\eta d\alpha d\xi d\beta$ is $4\epsilon^2$, we find that

$$
L_2 = \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^+}\int_{\mathbb{R}} |u(\eta,\alpha)-v(\eta,\alpha)|4\psi_t(\eta-\xi\epsilon,\alpha-\beta\epsilon)\rho(\beta)\rho(\xi)\,d\eta d\alpha d\xi d\beta.
$$

Since $\int_{-\infty}^{\infty}\rho(x)\,dx=\frac{1}{2}$, we obtain

$$
\lim_{\epsilon\to 0} L_2 = \int_{\mathbb{R}^+}\int_{\mathbb{R}} |u(\eta,\alpha)-v(\eta,\alpha)|\psi_t(\eta,\alpha)\,d\eta d\alpha.
$$

Therefore, the first integrand in (5.2.5) is reduced to

$$
\int_{\mathbb{R}^+}\int_{\mathbb{R}} |u(x,t)-v(x,t)|\psi_t(x,t)\,dxdt. \tag{5.2.7}
$$

Consider the second integrand in (5.2.5),

$$
\begin{aligned}
&\int_{\mathbb{R}^+}\int_{\mathbb{R}}\int_{\mathbb{R}^+}\int_{\mathbb{R}} (I_{1,1}+J_{1,1})\,dxdtd\bar{x}d\bar{t} \\
&= \int_{\mathbb{R}^+}\int_{\mathbb{R}}\int_{\mathbb{R}^+}\int_{\mathbb{R}} sgn(u(x,t)-v(\bar{x},\bar{t}))\big(F(u,\bar{u})-F(v,\bar{v})\big)(\partial_x+\partial_{\bar{x}})\phi\,dxdtd\bar{x}d\bar{t} \\
&= \int_{\mathbb{R}^+}\int_{\mathbb{R}}\int_{\mathbb{R}^+}\int_{\mathbb{R}} sgn(u(x,t)-v(\bar{x},\bar{t}))\big(F(u,\bar{u})-F(v,\bar{v})\big) \\
&\qquad\qquad \times\epsilon^{-2}\psi_x\Big(\frac{x+\bar{x}}{2},\frac{t+\bar{t}}{2}\Big)\rho\Big(\frac{t-\bar{t}}{2\epsilon}\Big)\rho\Big(\frac{x-\bar{x}}{2\epsilon}\Big)\,dxdtd\bar{x}d\bar{t}.
\end{aligned}
\tag{5.2.8}
$$

For this integral, we mimic the arguments of Lemma 3 and Theorem 1 of [Kružkov (1970)]. In fact, the integrand of (5.2.8) can be represented in the form

$$H(x,t,\bar{x},\bar{t},u(x,t),v(\bar{x},\bar{t}),\bar{u}(x,t),\bar{v}(\bar{x},\bar{t}))\epsilon^{-2}\rho\left(\frac{t-\bar{t}}{2\epsilon}\right)\rho\left(\frac{x-\bar{x}}{2\epsilon}\right),$$

where the function H satisfies a Lipschitz condition in all its variables. Consider

$$\begin{aligned}
&\int_{\mathbb{R}^+}\int_{\mathbb{R}}\int_{\mathbb{R}^+}\int_{\mathbb{R}} H(x,t,\bar{x},\bar{t},u(x,t),v(\bar{x},\bar{t}),\bar{u}(x,t),\bar{v}(\bar{x},\bar{t}))\epsilon^{-2}\rho\left(\frac{t-\bar{t}}{2\epsilon}\right)\rho\left(\frac{x-\bar{x}}{2\epsilon}\right)dxdtd\bar{x}d\bar{t}\\
&=\int_{\mathbb{R}^+}\int_{\mathbb{R}}\int_{\mathbb{R}^+}\int_{\mathbb{R}}\Big[H(x,t,\bar{x},\bar{t},u(x,t),v(\bar{x},\bar{t}),\bar{u}(x,t),\bar{v}(\bar{x},\bar{t}))\\
&\qquad\qquad - H(x,t,x,t,u(x,t),v(x,t),\bar{u}(x,t),\bar{v}(x,t))\Big]\epsilon^{-2}\rho(\,\cdot\,)\rho(\,\cdot\,)dxdtd\bar{x}d\bar{t}\\
&\qquad + \int_{\mathbb{R}^+}\int_{\mathbb{R}}\int_{\mathbb{R}^+}\int_{\mathbb{R}} H(x,t,x,t,u(x,t),v(x,t),\bar{u}(x,t),\bar{v}(x,t))\Big]\epsilon^{-2}\rho(\,\cdot\,)\rho(\,\cdot\,)dxdtd\bar{x}d\bar{t}\\
&=: L_3+L_4.
\end{aligned}$$

$$|L_3|\le C\int_{\mathbb{R}^+}\int_{\mathbb{R}}\int_{\mathbb{R}^+}\int_{\mathbb{R}}\Big[|x-\bar{x}|+|t-\bar{t}|+|v(x,t)-v(\bar{x},\bar{t})|+|\bar{v}(x,t)-\bar{v}(\bar{x},\bar{t})|\Big]\epsilon^{-2}\rho(\cdot)\rho(\cdot)dxdtd\bar{x}d\bar{t}. \tag{5.2.9}$$

It is obvious that the first and second integrals approach zero as ϵ approaches zero. The third integral approaches zero too, due to Lemma 2 in [Kružkov (1970)]. Since

$$\begin{aligned}
|\bar{v}(x,t)-\bar{v}(\bar{x},\bar{t})| &= \left|\int_{-\infty}^{\infty} K(y)v(x-y,t)-K(y)v(\bar{x}-y,\bar{t})\,dy\right|\\
&\le \|K\|_\infty\int_{-\infty}^{\infty}|v(x-y,t)-v(\bar{x}-y,\bar{t})|\,dy,
\end{aligned}\tag{5.2.10}$$

the last integral in (5.2.9) approaches zero as ϵ approaches zero.

Next, the integral L_4 can be shown to converge to

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}} H(x,t,x,t,u(x,t),v(x,t),\bar{u}(x,t),\bar{v}(x,t))\,dxdt.$$

Therefore,

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^+}\int_{\mathbb{R}}\int_{\mathbb{R}^+}\int_{\mathbb{R}}(I_{1,1}+J_{1,1})\,dxdtd\bar{x}d\bar{t}=\int_{\mathbb{R}^+}\int_{\mathbb{R}} sgn(u-v)\Big(F(u,\bar{u})-F(v,\bar{v})\Big)\partial_x\psi\,dxdt. \tag{5.2.11}$$

Next, consider the last term in (5.2.5),

$$I_{1,2} + J_{1,2} := I_2 = \underbrace{sgn(u-v)\Big[\big(F(v,\bar v) - F(v,\bar u)\big)\phi_x - \big(F(u,\bar u) - F(u,\bar v)\big)\phi_{\bar x}\Big]}_{I_{2,1}} + \underbrace{sgn(u-v)\Big(F(u,\bar v)_{\bar x} - F(v,\bar u)_x\Big)\phi}_{I_{2,2}}. \tag{5.2.12}$$

With the test function ϕ defined in (5.2.6), we find that

$$I_{2,1} = \underbrace{sgn(u-v)\Big[\big(F(v,\bar v) - F(v,\bar u)\big) - \big(F(u,\bar u) - F(u,\bar v)\big)\Big]\psi_x\epsilon^{-2}\rho\Big(\frac{t-\bar t}{2\epsilon}\Big)\rho\Big(\frac{x-\bar x}{2\epsilon}\Big)}_{I_{2,1,1}} + \underbrace{sgn(u-v)\Big[\big(F(v,\bar v) - F(v,\bar u)\big) + \big(F(u,\bar u) - F(u,\bar v)\big)\Big]\psi\epsilon^{-2}\rho\Big(\frac{t-\bar t}{2\epsilon}\Big)\rho_x\Big(\frac{x-\bar x}{2\epsilon}\Big)}_{I_{2,1,2}}. \tag{5.2.13}$$

We use integration by parts to show that

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}}\int_{\mathbb{R}^+}\int_{\mathbb{R}} I_{2,1,2}\; dxdtd\bar x d\bar t$$
$$= -\iiiint \underbrace{sgn(u-v)\Big[\big(F(v,\bar v) - F(v,\bar u)\big) + \big(F(u,\bar u) - F(u,\bar v)\big)\Big]_x \phi}_{I_{2,1,2,1}}$$
$$- \iiiint \underbrace{sgn(u-v)\Big[\big(F(v,\bar v) - F(v,\bar u)\big) + \big(F(u,\bar u) - F(u,\bar v)\big)\Big]\partial_x\psi\epsilon^{-2}\rho\Big(\frac{t-\bar t}{2\epsilon}\Big)\rho\Big(\frac{x-\bar x}{2\epsilon}\Big)}_{I_{2,1,2,2}}.$$

We introduce a non-negative function $\delta \in C_0^\infty$, satisfying

$$\delta(y) = \delta(-y), \quad \delta(y) = 0, \; for \; |\delta| \geq 1, \; and \; \int_{\mathbb{R}} \delta(y)\, dy = 1,$$

and set

$$\delta_\eta(t) = \frac{1}{\eta}\delta\Big(\frac{t}{\eta}\Big), \quad \epsilon > 0,$$

$$\chi_\eta(t) = \int_{-\infty}^{t} \Big(\delta_\eta(\tau - t_1) - \delta_\eta(\tau - t_2)\Big)\, d\tau, \quad 0 < t_1 < t_2 < \infty.$$

For $r > 1$ set

$$\varphi_r(x) = \int_{\mathbb{R}} \delta(|x-y|)\mathbf{1}_{|y|<r}\, dy,$$

so that

$$\partial_x\varphi_r(x) = 0, \quad for \; |x| < r-1 \; or \; |x| > r+1. \tag{5.2.14}$$

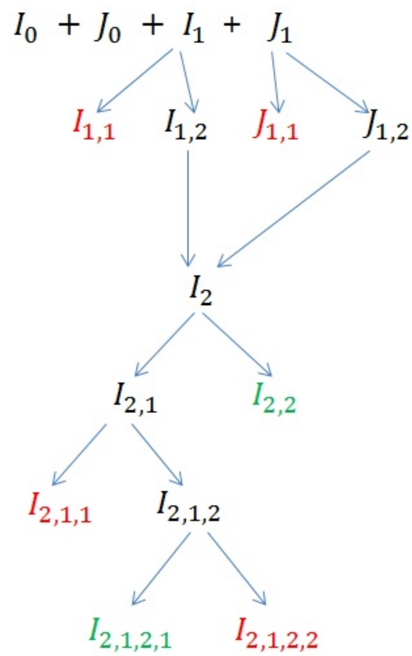

Figure 5.1 Decomposition diagram

We shall write (5.2.7) with $\psi(x,t) = \varphi_r(x)\chi_\eta(t)$, let $\eta \to 0$ and then let $r \to \infty$, so that (5.2.7) is reduced to

$$\|u(\cdot,t_2) - v(\cdot,t_2)\|_{L^1} - \|u(\cdot,t_1) - v(\cdot,t_1)\|_{L^1}.$$

The integrals $I_{1,1} + J_{1,1}$ in (5.2.11), $I_{2,1,1}$ in (5.2.13) and $I_{2,1,2,2}$ approach zero due to (5.2.14).

Consider the following remaining terms(see the diagram),

$$\begin{aligned} I_{2,2}+&I_{2,1,2,1} \\ &= sgn(u-v)\Big[F(u,\bar{v})_{\bar{x}} - F(v,\bar{u})_x + \Big(-F(v,\bar{v}) + F(v,\bar{u}) - F(u,\bar{u}) + F(u,\bar{v})\Big)_x\Big]\phi \\ &= sgn(u-v)\Big[F(u,\bar{v})_{\bar{x}} + \Big(-F(u,\bar{u}) + F(u,\bar{v})\Big)_x\Big]\phi. \end{aligned}$$

Therefore, (5.2.5) is reduced to

$$\begin{aligned} \|u(\cdot,t)-&v(\cdot,t)\|_{L^1}\Big|_{t_1}^{t_2} \\ &\le \underbrace{\lim_{\epsilon,\eta\to 0,\ r\to\infty} \iiiint sgn(u-v)\Big[F(u,\bar{v})_{\bar{x}} + \Big(-F(u,\bar{u}) + F(u,\bar{v})\Big)_x\Big]\phi\, dxdtd\bar{x}d\bar{t}}_{RHS}. \end{aligned}$$

Note that the integrand in the right hand side is

$$sgn(u-v)\Big[F^2(u,\bar{v})\bar{v}_{\bar{x}} - F^1(u,\bar{u})u_x - F^2(u,\bar{u})\bar{u}_x + F^1(u,\bar{v})u_x\Big]\phi.$$

After we use the mapping $x = \eta$, $t = \alpha$, $(t - \bar{t})/2\epsilon = \beta$, $(x - \bar{x})/2\epsilon = \xi$, the definitions of ϕ and ψ, we obtain

$$\begin{aligned} RHS &\leq \int_{t_1}^{t_2} \int_{\mathbb{R}} \Big| F^2(u,\bar{v})\bar{v}_x - F^1(u,\bar{u})u_x - F^2(u,\bar{u})\bar{u}_x + F^1(u,\bar{v})u_x \Big| \, dxdt \\ &\leq \int_{t_1}^{t_2} \int_{\mathbb{R}} \big| - F^1(u,\bar{u}) + F^1(u,\bar{v}) \big| |u_x| + \big| F^2(u,\bar{v})\bar{v}_x - F^2(u,\bar{u})\bar{u}_x \big| \, dxdt. \end{aligned} \tag{5.2.15}$$

We observe that

$$\big| - F^1(u,\bar{u}) + F^1(u,\bar{v}) \big| \leq \|F^{1,2}\|_\infty \|K\|_\infty \|u - v\|_{L^1},$$

$$\begin{aligned} \big| F^2(u,\bar{v})\bar{v}_x - F^2(u,\bar{u})\bar{u}_x \big| &\leq |F^2(u,\bar{v})\bar{v}_x - F^2(u,\bar{v})\bar{u}_x| + |F^2(u,\bar{v})\bar{u}_x - F^2(u,\bar{u})\bar{u}_x| \\ &\leq \|F^2\|_\infty |\bar{u}_x - \bar{v}_x| + |\bar{u}_x| |F^2(u,\bar{v}) - F^2(u,\bar{u})| \\ &\leq \|F^2\|_\infty |(K_x * (u - v))(x,t)| + |\bar{u}_x| \|F^{2,2}\|_\infty \|K\|_\infty \|u - v\|_{L^1}. \end{aligned}$$

Taking into account the fact that u has bounded variation and $\|K_x * (u-v)\|_{L^1} \leq \|K_x\|_{L^1} \|u - v\|_{L^1}$ we arrive at

$$\|u(\cdot,t_2) - v(\cdot,t_2)\|_{L^1} \leq \|u(\cdot,t_1) - v(\cdot,t_1)\|_{L^1} + C \int_{t_1}^{t_2} \|u(\cdot,s) - v(\cdot,s)\|_{L^1} \, ds.$$

Sending $t_1 \downarrow 0$ and setting $t_2 = t \leq T$, for any $T > 0$, upon using the Gronwall's inequality, concludes the proof of the theorem.

5.3 Existence of The Entropy Solution

Our task here is to construct entropy solutions of the non-local conservation law (5.1.1). Ever since the first fundamental paper [Hopf (1950)] was published, the main method for investigating quasilinear equations has been the vanishing viscosity method, which is based on the idea of passing to the limit as $\epsilon \to 0$ in the family of parabolic equations:

$$\begin{cases} \partial_t u(x,t) + \partial_x F(u,\bar{u}) = \epsilon \partial_x^2 u(x,t), \\ u(x,0) = u_0(x). \end{cases} \tag{5.3.1}$$

Because the first equation in (5.3.1) is parabolic, the initial value problem (5.3.1) admits a unique solution, which is smooth for $t > 0$ even when the initial data u_0 are only in L^∞, see e.g., [Kreiss et al. (2004); Lunardi (1995)].

We first show that the $\epsilon \to 0$ limit of solutions of (5.3.1) does satisfy the Kružkov inequality in Definition 5.1.2.

Theorem 5.3.1. *Let u_ϵ denote the solution of* (5.3.1)*. Assume that for some sequence $\{\epsilon_j\}$, with $\epsilon_j \downarrow 0$ as $j \to \infty$, $\{u_{\epsilon_j}\}$ converges to some function u. Then u is an entropy solution of* (5.1.1) *on $\mathbb{R} \times [0, \infty)$.*

Proof. Consider any smooth convex function η. Multiply (5.3.1) by $\eta'(u_\epsilon(x,t))$,

$$\partial_t \eta(u_\epsilon) + \eta'(u_\epsilon)\partial_x F(u_\epsilon, \bar{u}_\epsilon) = \epsilon \partial_x^2 \eta(u_\epsilon) - \epsilon \eta''(u_\epsilon)(\partial_x u_\epsilon)^2.$$

We multiply the above by nonnegative test function $\psi \in C_0^\infty(\mathbb{R} \times [0,\infty))$, integrate over $\mathbb{R} \times \mathbb{R}^+$, and integrate by parts.

$$\begin{aligned}\int_{\mathbb{R}^+}\int_{\mathbb{R}} \partial_t \psi \eta(u_\epsilon)\, dxdt - \int_{\mathbb{R}^+}\int_{\mathbb{R}} \psi \eta'(u_\epsilon)\partial_x F(u_\epsilon, \bar{u}_\epsilon)\, dxdt &= -\epsilon \int_{\mathbb{R}^+}\int_{\mathbb{R}} \psi_{xx}\eta(u_\epsilon)\, dxdt \\ &\quad + \epsilon \int_{\mathbb{R}^+}\int_{\mathbb{R}} \psi \eta''(u_\epsilon)(\partial_x u_\epsilon)^2\, dxdt.\end{aligned}$$

Since the last term is non-negative, we obtain

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}} \partial_t \psi \eta(u_\epsilon)\, dxdt - \int_{\mathbb{R}^+}\int_{\mathbb{R}} \psi \eta'(u_\epsilon)\partial_x F(u_\epsilon, \bar{u}_\epsilon)\, dxdt \geq -\epsilon \int_{\mathbb{R}^+}\int_{\mathbb{R}} \psi_{xx}\eta(u_\epsilon)\, dxdt. \quad (5.3.2)$$

Consider the second term in (5.3.2), by the integration by parts,

$$\begin{aligned}-\int_{\mathbb{R}^+}\int_{\mathbb{R}} \psi \eta'(u_\epsilon)\partial_x F(u_\epsilon, \bar{u}_\epsilon)\, dxdt &= -\int_{\mathbb{R}^+}\int_{\mathbb{R}} \psi \eta'(u_\epsilon)\partial_x \big[F(u_\epsilon, \bar{u}_\epsilon) + F(k, \bar{u}_\epsilon) - F(k, \bar{u}_\epsilon)\big]\, dxdt \\ &= \int_{\mathbb{R}^+}\int_{\mathbb{R}} \psi_x \eta'(u_\epsilon) F(u_\epsilon, \bar{u}_\epsilon) + \psi \partial_x \eta'(u_\epsilon) F(u_\epsilon, \bar{u}_\epsilon)\, dxdt \\ &\quad - \int_{\mathbb{R}^+}\int_{\mathbb{R}} \psi \eta'(u_\epsilon) F(k, \bar{u}_\epsilon)_x\, dxdt \\ &\quad - \int_{\mathbb{R}^+}\int_{\mathbb{R}} \psi_x \eta'(u_\epsilon) F(k, \bar{u}_\epsilon) + \psi \partial_x \eta'(u_\epsilon) F(k, \bar{u}_\epsilon)\, dxdt.\end{aligned} \quad (5.3.3)$$

Now we let $\eta(u_\epsilon) := |u_\epsilon - k|$, then $\eta'(u_\epsilon) = sgn(u_\epsilon - k)$. And since $\eta''(u_\epsilon) = \delta(u_\epsilon - k)$, we notice that

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}} \psi \partial_x \eta'(u_\epsilon)\big[F(u_\epsilon, \bar{u}_\epsilon) - F(k, \bar{u}_\epsilon)\big]\, dxdt = 0.$$

Indeed,

$$\begin{aligned}&\left| \iint \psi \partial_x \eta'(u_\epsilon)\big[F(u_\epsilon, \bar{u}_\epsilon) - F(k, \bar{u}_\epsilon)\big]\, dxdt \right| \\ &\qquad = \left| \iint \psi(x,s)\delta(u_\epsilon - k) u_{\epsilon x}\big[F(u_\epsilon, \bar{u}_\epsilon) - F(k, \bar{u}_\epsilon)\big]\, dxdt \right| \\ &\qquad \leq \int_{\mathbb{R}^+} \|\psi(\cdot, s)\|_{L^\infty} \|F^1\|_{L^\infty} \int_{\mathbb{R}} \delta(u_\epsilon(x,s) - k)|u_{\epsilon x}||u_\epsilon(x,s) - k|\, dxdt.\end{aligned}$$

For any fixed t, we have

$$\int_{\mathbb{R}} \delta(u_\epsilon(x,s)-k)|u_{\epsilon x}||u_\epsilon(x,s)-k|\,dx = \int_{Range(u_\epsilon(\cdot,s)-k)} \delta(y)|y|\,dy = 0.$$

Therefore, the right hand side of (5.3.3) is reduced to

$$\begin{aligned}\int_{\mathbb{R}^+}\int_{\mathbb{R}} \psi_x sgn(u_\epsilon - k)F(u_\epsilon,\bar{u}_\epsilon)dxdt - &\int_{\mathbb{R}^+}\int_{\mathbb{R}} \psi sgn(u_\epsilon - k)F(k,\bar{u})_x dxdt \\ &- \int_{\mathbb{R}^+}\int_{\mathbb{R}} \psi_x sgn(u_\epsilon - k)F(k,\bar{u}_\epsilon)dxdt.\end{aligned}$$

Substituting the above into (5.3.2), setting $\epsilon = \epsilon_j$ and letting $j \to \infty$, we conclude that the limit u satisfies (5.1.3). $\square$

Lemma 5.3.2. *Let $BV(u_0) := \|\partial_x u_0\|_{L^1(\mathbb{R})} < \infty$ and u_ϵ denote the solution of* (5.3.1)*. Then, for any $t > 0$, there exists a constant C such that*

$$\|\partial_x u_\epsilon(\cdot,t))\|_{L^1(\mathbb{R})} \leq e^{Ct}\|\partial_x u_\epsilon(\cdot,0)\|_{L^1(\mathbb{R})}.$$

Here, C depends only on u_0, F and kernel K.

Proof. For notational convenience, we omit the subscript ϵ from u_ϵ. We first choose a fixed function $Sgn(y) \in C^1(\mathbb{R})$ with

$$Sgn(y) = \begin{cases} -1, & y \leq -1 \\ 0, & y = 0 \\ 1, & y \geq 1, \end{cases}$$

and

$$Sgn'(y) \geq 0,$$

for all $y \in \mathbb{R}$. Then we set

$$sgn_\delta(y) = Sgn\left(\frac{y}{\delta}\right),\ y \in \mathbb{R},\ \delta > 0.$$

Now we find that

$$\frac{d}{dt}\int_{\mathbb{R}} sgn_\delta(u_x)u_x\,dx = \int_{\mathbb{R}} sgn'_\delta(u_x)u_{xt}u_x\,dx + \int_{\mathbb{R}} sgn_\delta(u_x)u_{xt}\,dx. \tag{5.3.4}$$

Here,

$$\begin{aligned}\int_{\mathbb{R}} sgn_\delta(u_x)u_{xt}\,dx &= \int_{\mathbb{R}} sgn_\delta(u_x)(\epsilon u_{xxx} - \partial_x^2 F(u,\bar{u}))\\ &= -\epsilon\int_{\mathbb{R}} sgn'_\delta(u_x)u_{xx}^2\,dx + \int_{\mathbb{R}} sgn'_\delta(u_x)u_{xx}\partial_x F(u,\bar{u})\,dx\\ &\leq \int_{\mathbb{R}} sgn'_\delta(u_x)u_{xx}\partial_x F(u,\bar{u})\,dx.\end{aligned}$$

Therefore

$$\frac{d}{dt}\int_{\mathbb{R}} sgn_\delta(u_x)u_x\,dx \leq \int_{\mathbb{R}} sgn'_\delta(u_x)\{u_{xt}u_x + u_{xx}\partial_x F(u,\bar{u})\}\,dx. \tag{5.3.5}$$

After expanding and rearranging the right hand side we obtain

$$\int_{\mathbb{R}} sgn'_\delta(u_x)u_x\{u_{xt} + u_{xx}F^1(u,\bar{u})\}\,dx - \int_{\mathbb{R}} sgn'_\delta(u_x)u_{xx}F^2(u,\bar{u})\bar{u}_x\,dx. \tag{5.3.6}$$

Note that $sgn'_\delta(u_x)u_x$ is bounded independently of δ and x. Therefore, Lebesque's Dominated Convergence Theorem yields

$$\int_{\mathbb{R}} sgn'_\delta(u_x)u_x\{u_{xt} + u_{xx}F^1(u,\bar{u})\}\,dx \to 0 \quad \text{as} \quad \delta \to 0.$$

For the second term in (5.3.6), we claim that

$$\left|\int_{\mathbb{R}} sgn'_\delta(u_x)u_{xx}F^2(u,\bar{u})\bar{u}_x\,dx\right| \leq C\|u_x(\cdot,t)\|_{L^1(\mathbb{R})}, \tag{5.3.7}$$

where C depends only on u_0, F and kernel K. Indeed, note that

$$\begin{aligned}\int_{\mathbb{R}} sgn'_\delta(u_x)u_{xx}F^2(u,\bar{u})\bar{u}_x\,dx &= \int_{\mathbb{R}} \partial_x\big(sgn_\delta(u_x)\big)F^2(u,\bar{u})\bar{u}_x\,dx\\ &= \int_{\mathbb{R}} sgn_\delta(u_x)\big\{F^{2,1}u_x\bar{u}_x + F^{2,2}\bar{u}_x^2 + F^2\bar{u}_{xx}\big\}\,dx.\end{aligned}$$

Applying obvious estimates $\|\bar{u}_x\|_\infty \leq \|u\|_\infty\|K_x\|_{L^1} \leq \|u_0\|_\infty\|K_x\|_{L^1}$, $\|\bar{u}_x\|_{L^1} \leq \|K\|_{L^1}\|u_x\|_{L^1}$, $\|\bar{u}_{xx}\|_{L^1} \leq \|K_x\|_{L^1}\|u_x\|_{L^1}$ and $|sgn_\delta(\cdot)| \leq 1$ yield the inequality in (5.3.7).

Finally, (5.3.5) is reduced to

$$\frac{d}{dt}\|u_x(\cdot,t)\|_{L^1(\mathbb{R})} \leq C\|u_x(\cdot,t)\|_{L^1(\mathbb{R})},$$

as $\delta \to 0$ and the Gronwall's inequality gives the desired result. $\square$

Theorem 5.3.3. *Let u_ϵ and v_ϵ be solutions of* (5.3.1) *with respective initial data u_0 and v_0 that are in $L^1(\mathbb{R}) \cap BV(\mathbb{R})$. Then, for any $t > 0$, there exists a constant C such that*

$$\|u_\epsilon(\cdot,t) - v_\epsilon(\cdot,t)\|_{L^1(\mathbb{R})} \leq \exp(\exp(Ct) - 1)\|u_0 - v_0\|_{L^1(\mathbb{R})}.$$

Here, C depends only on u_0, F and kernel K.

Proof. For notational convenience, we omit the subscript ϵ from u_ϵ and v_ϵ. Let

$$\eta_\mu(y) = \begin{cases} 0, & -\infty < y \leq 0, \\ \frac{y^2}{4\mu}, & 0 < y \leq 2\mu, \\ y - \mu, & 2\mu < y < \infty. \end{cases}$$

Since both u, v satisfy the first equation of (5.3.1), then $w = u - v$ solves

$$\partial_t \eta_\mu(w) + \eta'_\mu(w)\partial_x\big\{F(u,\bar{u}) - F(v,\bar{v})\big\} = \epsilon\partial_x^2\eta_\mu(w) - \epsilon\eta''_\mu(w)w_x^2.$$

Sine the last term is non-positive, we have

$$\partial_t \eta_\mu(w) + \eta'_\mu(w)\partial_x\big\{F(u,\bar{u}) - F(v,\bar{v})\big\} \leq \epsilon\partial_x^2\eta_\mu(w).$$

Integration over $\mathbb{R}$ yields

$$\int_{\mathbb{R}} \partial_t \eta_\mu(w)\,dx + \int_{\mathbb{R}} \eta'_\mu(w)\partial_x\big\{F(u,\bar{u}) - F(v,\bar{v})\big\}\,dx \leq 0. \tag{5.3.8}$$

We decompose the second integral into

$$\int_{\mathbb{R}} \eta'_\mu(w)\partial_x\big\{F(u,\bar{u}) - F(v,\bar{u}) + F(v,\bar{u}) - F(v,\bar{v})\big\}\,dx,$$

upon integration by parts gives

$$\int_{\mathbb{R}} \eta''_\mu(w)w_x\big\{F(u,\bar{u}) - F(v,\bar{u})\big\}\,dx + \int_{\mathbb{R}} \eta'_\mu(w)\partial_x\big\{F(v,\bar{u}) - F(v,\bar{v})\big\}\,dx. \tag{5.3.9}$$

Notice that $\eta''_\mu(u(x,t) - v(x,t))\big\{F(u,\bar{u}) - F(v,\bar{u})\big\}$ is bounded, uniformly for $\mu > 0$. Also it converges pointwise to zero. Therefore, the Lebesque dominated convergence theorem implies

$$\lim_{\mu\to 0} \int_{\mathbb{R}} \eta''_\mu(w)w_x\big\{F(u,\bar{u}) - F(v,\bar{u})\big\}\,dx = 0. \tag{5.3.10}$$

Consider the second integral in (5.3.9),

$$\begin{aligned}
&\int_{\mathbb{R}} \eta'_\mu(w)\big\{F^2(v,\bar{u})\bar{u}_x - F^2(v,\bar{v})\bar{v}_x + F^1(v,\bar{u})v_x - F^1(v,\bar{v})v_x\big\}\,dx \\
&= \underbrace{\int_{\mathbb{R}} \eta'_\mu(w)\big\{F^2(v,\bar{u}) - F^2(v,\bar{v})\big\}\bar{u}_x\,dx}_{J_1} + \underbrace{\int_{\mathbb{R}} \eta'_\mu(w)F^2(v,\bar{v})(\bar{u}_x - \bar{v}_x)\,dx}_{J_2} \\
&+ \underbrace{\int_{\mathbb{R}} \eta'_\mu(w)\big\{F^1(v,\bar{u}) - F^1(v,\bar{v})\big\}v_x\,dx}_{J_3}.
\end{aligned} \tag{5.3.11}$$

We claim that

$$|J_1 + J_2 + J_3| \le C\|u(\cdot,t) - v(\cdot,t)\|_{L^1}.$$

Indeed, the mean value property and the fact that

$$|\eta'_\mu(\cdot)| \le 1$$

and

$$\|\bar{u}_x\|_\infty \le \|u\|_\infty\|K_x\|_{L^1} \le \|u_0\|_\infty\|K_x\|_{L^1}$$

give

$$|J_1| \le C_1\int_{\mathbb{R}} |\bar{u} - \bar{v}|\,dx = C_1\int_{\mathbb{R}} |K*(u-v)(x)|\,dx \le C_1\|K\|_{L^1}\|u(\cdot,t) - v(\cdot,t)\|_{L^1},$$

$$|J_2| \le C_2\int_{\mathbb{R}} |\bar{u}_x - \bar{v}_x|\,dx = C_2\|K_x\|_{L^1}\|u(\cdot,t) - v(\cdot,t)\|_{L^1}, \text{ and}$$

$$|J_3| \le C_3\int_{\mathbb{R}} |\bar{u} - \bar{v}||v_x|\,dx \le C_3\|\bar{u} - \bar{v}\|_\infty\int_{\mathbb{R}} |v_x|\,dx \le C_3\|K\|_\infty\|u(\cdot,t) - v(\cdot,t)\|_{L^1}e^{C_4t}BV(v_0),$$

where we have used Lemma 5.3.2.

Hence, as $\mu \to 0$, (5.3.8) reduces to

$$\partial_t\int_{\mathbb{R}} [(u(x,t) - v(x,t)]^+\,dx \le Ce^{Ct}\|u(\cdot,t) - v(\cdot,t)\|_{L^1}.$$

Interchanging the roles of u and v we derive a similar inequality which added to the above yields

$$\partial_t\|u(\cdot,t) - v(\cdot,t)\|_{L^1} \le Ce^{Ct}\|u(\cdot,t) - v(\cdot,t)\|_{L^1}.$$

Finally, the Gronwall's inequality gives the desired result. □

Lemma 5.3.4. *Let u_ϵ be the solution to* (5.3.1), *with $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$. In particular,*

$$\int_{\mathbb{R}} |u_0(x+y) - u_0(x)|\, dx \leq \omega(|y|), \quad y \in \mathbb{R}, \tag{5.3.12}$$

for some nondecreasing function ω on $[0,\infty)$, with $\omega(r) \searrow 0$ as $r \searrow 0$. Then, for any $t > 0$,

i)

$$\int_{\mathbb{R}} |u_\epsilon(x+y,t) - u_\epsilon(x,t)|\, dx \leq \exp(\exp(Ct) - 1)\omega(|y|), \quad y \in \mathbb{R}. \tag{5.3.13}$$

ii)

$$\int_{\mathbb{R}} |u_\epsilon(x,t+h) - u_\epsilon(x,t)|\, dx \leq c(h^{2/3} + \epsilon h^{1/3})\|u_0\|_{L^1} + 2\exp(\exp(Ct) - 1)\omega(h^{1/3}), \quad h > 0. \tag{5.3.14}$$

Here, C depends only on u_0, F and kernel K.

Proof. i) Since $u_\epsilon(x,t)$ is the solution of (5.3.1), $u_\epsilon(x+y,t)$ is the solution of (5.3.1) with initial data $u_0(x+y)$, hence Theorem 5.3.3 yields,

$$\int_{\mathbb{R}} |u_\epsilon(x+y,t) - u_\epsilon(x,t)|\, dx \leq \exp(\exp(Ct) - 1) \int_{\mathbb{R}} |u_0(x+y) - u_0(x)|\, dx.$$

This completes the proof of (5.3.13).

ii) We multiply (5.3.1) by $\phi(x) \in C_0^\infty(\mathbb{R})$ and integrate over $\mathbb{R} \times (t, t+h)$ to obtain,

$$\int_{\mathbb{R}} \phi(x)\big[u_\epsilon(x,t+h) - u_\epsilon(x,t)\big]\, dx = \int_t^{t+h} \int_{\mathbb{R}} \big[F(u,\bar{u})\phi_x + \epsilon u \phi_{xx}\big]\, dxds. \tag{5.3.15}$$

Let $v(x) := u_\epsilon(x,t+h) - u_\epsilon(x,t)$, and

$$\phi(x) = \int_{-\infty}^{\infty} h^{-1/3} \rho\left(\frac{x-y}{h^{1/3}}\right) sgn(v(y))\, dy,$$

where ρ is a smooth and non-negative function with support contained in $[-1,1]$ and

$$\int_{-1}^{1} \rho(y)\, dy = 1.$$

Then, $|\phi_x| \leq c_1 h^{-1/3}$ and $|\phi_{xx}| \leq c_2 h^{-2/3}$. Since $\|u(\cdot,s)\|_\infty \leq \|u_0\|_\infty$, then from (5.3.15), it follows

$$\int_{\mathbb{R}} \phi(x) v(x)\, dx \leq c(h^{2/3} + \epsilon h^{1/3})\|u_0\|_\infty. \tag{5.3.16}$$

Here, c depends only on the range of $u_0(x)$. On the other hand, we observe that

$$\begin{aligned} |v(x)| - \phi(x)v(x) &= \int_{\mathbb{R}} h^{-1/3}\rho\left(\frac{x-y}{h^{1/2}}\right)\left[|v(x)| - v(x)sgn(v(y))\right] dy \\ &\leq 2\int_{\mathbb{R}} h^{-1/3}\rho\left(\frac{x-y}{h^{1/2}}\right)|v(x)-v(y)|\, dy, \end{aligned}$$

where the last inequality holds because of

$$|v(x)| - v(x)sgn(v(y)) \leq 2|v(x) - v(y)|.$$

Then, by a change of variable, we obtain

$$|v(x)| - \phi(x)v(x) \leq 2\int_{|z|<1} \rho(z)|v(x) - v(x - h^{1/3}z)|\, dz.$$

Integrating the above over $\mathbb{R}$ and combining with (5.3.16) and (5.3.13), we obtain (5.3.14). $\square$

We have now laid the groundwork for presenting an existence result.

Assume first that $u_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}) \cap BV(\mathbb{R})$. Let u_ϵ denote the solution of (5.3.1), with $0 < \epsilon < 1$. By Theorem 5.3.3 and Lemma 5.3.4 the family $\{u_\epsilon\}$ is uniformly bounded and equicontinuous in the mean on any compact subset of $\mathbb{R} \times (0, \infty)$. Consequently, by virtue of the Fréchet-Kolmogorov theorem, any sequence $\{\epsilon_k\}$, with $\epsilon_k \to$ as $k \to \infty$, will contain a subsequence, denoted again by $\{\epsilon_k\}$, such that $\{u_{\epsilon_k}\}$ converges to some function u. On account of Theorem 5.3.1, u is an entropy solution to (5.1.1). By Theorem 5.1.3, since the entropy solution is unique, the whole family $\{u_\epsilon\}$ must converge to u, as $\epsilon \to 0$. This completes the proof of the existence result.

CHAPTER 6. SUMMARY AND DISCUSSION

6.1 General Conclusion

Our investigation has been on the persistence of the C^1 solution regularity for Euler-Poisson equations and a non-local conservation laws. Our effort is to find an answer to the question of whether there is a critical threshold for the initial data such that the persistence of the C^1 solution regularity depends only on crossing such a critical threshold.

Our main achievements in the theory of critical thresholds are: i) identifying both upper-thresholds for the finite-time blow up and sub-thresholds for the global existence of solutions to restricted Euler-Poisson equations. ii) identifying upper-thresholds for finite time shock formation of a large class of non-local conservation laws.

6.2 Future Work

We plan to continue our investigation on the nonlocal conservation laws of the form

$$\begin{cases} \partial_t u + \sum_{i=1}^d \partial_{x_i} F(u, \bar{u}^i) = 0, & t > 0,\, x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d, \end{cases}$$

where $\bar{u}^i = \partial_{x_i} K * u$.

Our immediate goals are two fold: i) to establish well-posedness of entropy solutions for a class of multi-dimensional nonlocal conservation laws; ii) to identify sub-thresholds for global existence and upper-thresholds for finite time shock formation for multi-dimensional nonlocal conservation laws. One interesting application of this class is the hyperbolic Keller-Segel model [Yasmin et al. (2005)],

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho(1-\rho)\nabla S) = 0, \\ \Delta S = S - \rho, \end{cases}$$

where ρ is the cell density and S is the chemo-attractant. There are several new difficulties to be attacked: i) even in one-dimensional case, the interaction kernel is symmetric, different from that in the traffic flow model with looking ahead relaxation; ii) There is more coupling from different directions in multi-dimensional case, making it harder to control the solution gradient.

In the future, we plan to work along this line and hope to identify some biologically meaningful quantity that can be used in threshold analysis.

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